
Failure Modes of Variational Autoencoders and Their Effects on Downstream Tasks

Yaniv Yacoby, Weiwei Pan, Finale Doshi-Velez
John A. Paulson School of Engineering and Applied Sciences
Harvard University
Cambridge, MA 02138
{yanivyacoby, weiweipan}@g.harvard.edu, finale@seas.harvard.edu

Abstract

Variational Auto-encoders (VAEs) are deep generative latent variable models that are widely used for a number of downstream tasks. While it has been demonstrated that VAE training can suffer from a number of pathologies, existing literature lacks characterizations of exactly when these pathologies occur and how they impact down-stream task performance. In this paper we concretely characterize conditions under which VAE training exhibits pathologies and connect these failure modes to undesirable effects on specific downstream tasks – learning compressed and disentangled representations, adversarial robustness and semi-supervised learning.

1 Introduction

Variational Auto-encoders (VAEs) are deep generative latent variable models that transform simple distributions over a latent space to model complex data distributions [25]. They have been used for a wide range of downstream tasks, including: generating realistic looking synthetic data (e.g [38]), learning compressed representations (e.g. [35, 13, 2]), adversarial defense using de-noising [31, 12], and, when expert knowledge is available, generating counter-factual data using weak or semi-supervision (e.g. [23, 43, 26]). Formally, a VAE has two components: a *generative* model that transforms a distribution over latents $p(z)$ into a distribution over data $p(x)$, and an amortized *inference* model that provides an approximate posterior $q(z|x) \approx p(z|x)$. Both generative and inference models are usually parameterized as neural networks and are trained jointly to maximize a lower bound to the likelihood of the observed data. One reason for the popularity of VAE is the relative simplicity of its training procedure. In particular, the common choice of mean-field Gaussian (MFG) approximate posteriors for VAEs (MFG-VAE) results an inference procedure that is straight-forward to implement and stable in training.

Unfortunately, a growing body of work has demonstrated that MFG-VAEs suffer from a variety of pathologies, including learning un-informative latent codes (posterior collapse) (e.g.[49, 20]) and unrealistic data distributions (mismatch between aggregate posterior and prior) (e.g. [33, 47]). Recent work [52] attributes a number of these pathologies to properties of the training objective; in particular, the objective may compromise learning a good generative model in order to learn a good inference model – in other words, the inference model over-regularizes the generative model. While this pathology has been noted in literature [4, 53, 6], no prior work has characterizes the conditions under which the MFG-VAE objective compromises learning a good generative model in order to learn a good inference model; more worryingly, no prior work has related MFG-VAE pathologies with the performance of MFG-VAEs on downstream tasks. Rather, existing literature focuses on mitigating the regularizing effect of the inference model on the VAE generative model by using richer variational families (e.g. [24, 7, 36, 32]). While promising, these methods introduce potentially significant additional computational costs to training, as well as new training issues (e.g. noisy gradients [40, 48]). As such, it is important to understand precisely when MFG-VAEs

exhibit pathologies and when alternative training methods (e.g. ones using richer variational families) are worth the computational trade-off. In this paper, we characterize the conditions under which MFG-VAEs perform poorly and link them directly to their performance on a variety of downstream tasks.

Our contributions are theoretical and empirical: (1) We characterize concrete conditions under which learning the inference model will compromise learning the generative model for MFG-VAEs. More problematically, we show that these bad solutions are globally optimal for the training objective, the ELBO. (2) We demonstrate that using the ELBO to select the output noise variance and the latent dimension results in biased estimates. (3) Furthermore, we demonstrate ways in which these pathologies affects key downstream tasks, including learning compressed and disentangled representations, adversarial robustness and semi-supervised learning. (4) Lastly, we show that while the use of richer variational families alleviate VAE pathologies on unsupervised learning tasks, they introduce new ones in the semi-supervised tasks.

2 Related Work

Existing work that characterize MFG-VAEs pathologies primarily focus on relating local optima of the training objective to a single pathology: the un-informativeness of the learned latent codes (posterior collapse) [14, 30, 9]. In contrast, there has been little work to characterize pathologies at the global optima of the MFG-VAEs training objective. [52] shows that, when the decoder’s capacity is restricted, posterior collapse and the mismatch between aggregated posterior and prior can occur as global optima of the training objective. In contrast to existing work, we focus on global optima of the MFG-VAE objective in fully general settings: with fully flexible generative and inference models, as well as with and without learned observation noise.

While works focused on improving VAEs have noted the over-regularizing effect of the variational family on the generative model (e.g. [4, 53, 6]), none have given a full characterization of the conditions under which the learned generative model is meaningfully compromised. Nor have they related the resulting bias in VAE training to potentially impactful effects on down-stream tasks. In particular, these works have shown that their proposed methods have higher test log-likelihood relative to a MFG-VAEs, but as we show in this paper, high test log-likelihood is not the only property needed for good performance on downstream tasks. Lastly, these works all propose fixes that require a potentially significant computational overhead. For instance, works that use complex variational families, such as normalizing flows [24], require a significant number of parameters to scale [22]. As in the case of the Importance Weighted Autoencoder (IWAE) objective [4], which can be interpreted as having a more complex variational family [7], the complexity of the posterior scales with the number of importance samples used. Lastly, works that de-bias or reduce the variance of existing bounds [36, 32, 48, 40] all require several evaluations of the objective.

Given that MFG-VAEs remain popular today due to the ease of their implementation, speed of training, and their theoretical connections to other dimensionality reduction approaches like probabilistic PCA [41, 8, 30], it is important to characterize the training pathologies of MFG-VAE, as well as the concrete connections between these pathologies and down-stream tasks. More importantly, this characterization will help clarify for which tasks / datasets a MFG-VAE suffices and for which the computational tradeoffs are worth it.

3 Background

Unsupervised VAEs [25] In this paper, we assume N number of observations of x in \mathbb{R}^D . A VAE assumes the following generative process:

$$p(z) = \mathcal{N}(0, I), \quad p_\theta(x|z) = \mathcal{N}(f_\theta(z), \sigma_\epsilon^2 \cdot I) \quad (1)$$

where $z \in \mathbb{R}^K$ is a latent variable and f_θ is a neural network parametrized by θ . We learn the likelihood parameters θ while jointly approximating the posterior $p_\theta(z|x)$ with $q_\phi(z|x)$:

$$\max_{\theta} \mathbb{E}_{p(x)} [\log p_\theta(x)] \geq \max_{\theta, \phi} \mathbb{E}_{p(x)} \left[\mathbb{E}_{q_\phi(z|x)} \left[\log \frac{p_\theta(x|z)p(z)}{q_\phi(z|x)} \right] \right] = \text{ELBO}(\theta, \phi) \quad (2)$$

where $p(x)$ is the empirical data distribution, $p_\theta(x)$ is the learned data distribution, and $q_\phi(z|x)$ is a MFG with mean and variance $\mu_\phi(x), \sigma_\phi^2(x)$. We assume that $\mu_\phi(x)$ and $\sigma_\phi^2(x)$ are outputs of a neural network with parameters ϕ .

The VAE ELBO can alternately be written as a sum of two objectives – the ‘‘MLE objective’’ (MLEO), which maximizes the $p_\theta(x)$, and the ‘‘posterior matching objective’’ (PMO), which encourages variational posteriors to match posteriors of the generative model. That is, we can write:

$$\operatorname{argmin}_{\theta, \phi} - \text{ELBO}(\theta, \phi) = \operatorname{argmin}_{\theta, \phi} \underbrace{\left[D_{\text{KL}}[p(x) || p_\theta(x)] \right]}_{\text{MLEO}} + \underbrace{\mathbb{E}_{p(x)} \left[D_{\text{KL}}[q_\phi(z|x) || p_\theta(z|x)] \right]}_{\text{PMO}} \quad (3)$$

This decomposition allows for a more intuitive interpretation of VAE training and illustrates the tension between approximating the true posteriors and approximating $p(x)$.

Semi-Supervised VAEs We extend VAE model and inference to incorporate partial labels, allowing for some supervision of the latent space dimensions. For this, we use the semi-supervised model first introduced in [23] as the ‘‘M2 model’’. We assume the following generative process:

$$z \sim \mathcal{N}(0, I), \quad \epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2 \cdot I), \quad y \sim p(y), \quad x|y, z = f_\theta(y, z) + \epsilon \quad (4)$$

where y is observed only a portion of the time. Inference objective for this model can be written as a sum of two objectives, a lower bound for the likelihood of M labeled observations and a lower bound for the likelihood for N unlabeled observations:

$$\mathcal{J}(\theta, \phi) = \sum_{n=1}^N \mathcal{U}(x_n; \theta, \phi) + \gamma \cdot \sum_{m=1}^M \mathcal{L}(x_m, y_m; \theta, \phi) \quad (5)$$

where the \mathcal{U} and \mathcal{L} lower bound $p_\theta(x)$ and $p_\theta(x, y)$, respectively (see Appendix A for full definition), and γ controls their relative weight (as done in [44]).

Following [23], we assume a MFG variational family for each of the unlabeled and labeled objectives: $q_\phi(z|x) = \mathcal{N}(\mu_\phi(x), \sigma_\phi^2(x) \cdot I)$, $q_\phi(z, y|x) = q_\phi(y) \cdot \mathcal{N}(\mu_\phi(x), \sigma_\phi^2(x) \cdot I)$, where $q_\phi(y) = \text{Cat}(\pi_\phi(x))$ when $p(y)$ is categorical and $q_\phi(y) = \mathcal{N}(m_\phi(x), s_\phi^2(x) \cdot I)$ when $p(y) = \mathcal{N}(0, I)$. In [23, 44], an additional term is appended to the objective in Equation 11 in order to explicitly increase discriminative power of the posteriors $q_\phi(y_m|x_m)$:

$$\mathcal{J}^\alpha(\theta, \phi) = \mathcal{J}(\theta, \phi) + \alpha \cdot \sum_{m=1}^M \log q_\phi(y_m|x_m) \quad (6)$$

We will see in 5.2 that the additional term introduces unexpected training pathologies.

4 Characterizing Pathologies of the VAE Training Objective

In Section 4.1 we identify two pathological properties of the VAE training objective. In Section 4.2 we provide empirical demonstrations where these pathologies manifest on datasets and in Section 5.1 we unpack how these pathologies affect a variety of downstream tasks.

4.1 Pathologies of the VAE Objective

We fix a set of feasible likelihood functions \mathcal{F} and a variational family \mathcal{Q} , implied by our choice of the generative and inference model network architectures. We assume that \mathcal{F} is expressive enough to contain any smooth function, including the ground truth generating function, and we assume \mathcal{Q} contains all $q_{\phi^*}(z|x)$ corresponding to any $f_\theta \in \mathcal{F}$, where $\phi^* = \operatorname{argmin}_\phi - \text{ELBO}(\theta, \phi)$.

Pathology I: The ELBO trades off generative model quality for simple posteriors In the following, we show that the global optima of the VAE objective correspond to incorrect generative models under the following two conditions: (1) the true posterior is difficult to approximate by a MFG for a large portion of x ’s, and (2) there does not exist a likelihood function f_θ in \mathcal{F} with a simpler posterior that approximates $p(x)$ well.

We formalize these conditions in the following. Recall the decomposition the negative ELBO in Equation 3. In the following discussion, we always set ϕ to be optimal for our choice of θ . Assuming

that $p(x)$ is continuous, then for any $\eta \in \mathbb{R}$, we can further decompose the PMO:

$$\begin{aligned} \mathbb{E}_{p(x)} [D_{\text{KL}}[q_\phi(z|x)||p_\theta(z|x)]] = & \Pr[\mathcal{X}_{\text{Lo}}(\theta)] \mathbb{E}_{p(x)|\mathcal{X}_{\text{Lo}}} [D_{\text{KL}}[q_\phi(z|x)||p_\theta(z|x)]] \\ & + \Pr[\mathcal{X}_{\text{Hi}}(\theta)] \mathbb{E}_{p(x)|\mathcal{X}_{\text{Hi}}} [D_{\text{KL}}[q_\phi(z|x)||p_\theta(z|x)]] \end{aligned} \quad (7)$$

where $D_{\text{KL}}[q_\phi(z|x)||p_\theta(z|x)] \leq \eta$ on $\mathcal{X}_{\text{Lo}}(\theta)$, $D_{\text{KL}}[q_\phi(z|x)||p_\theta(z|x)] > \eta$ on $\mathcal{X}_{\text{Hi}}(\theta)$, with $\mathcal{X}_i(\theta) \subseteq \mathcal{X}$; where $\mathbb{E}_{p(x)|\mathcal{X}_i}$ is the expectation over $p(x)$ restricted to $\mathcal{X}_i(\theta)$ and renormalized, and $\Pr[\mathcal{X}_i]$ is the probability of $\mathcal{X}_i(\theta)$ under $p(x)$. Let us denote the expectation in first term on the right hand side of Equation 7 as $D_{\text{Lo}}(\theta)$ and the expectation in the second term as $D_{\text{Hi}}(\theta)$.

Let $f_{\theta_{\text{GT}}} \in \mathcal{F}$ be the ground truth likelihood function, for which we may assume that the MLE objective (MLEO) term is zero. We can now state our claim:

Theorem 1. *Suppose that there exist an $\eta \in \mathbb{R}$ such that $\Pr[\mathcal{X}_{\text{Hi}}(\theta_{\text{GT}})] D_{\text{Hi}}(\theta_{\text{GT}})$ is greater than $\Pr[\mathcal{X}_{\text{Lo}}(\theta_{\text{GT}})] D_{\text{Lo}}(\theta_{\text{GT}})$. Suppose that (1) there exist an $f_\theta \in \mathcal{F}$ such that $D_{\text{Lo}}(\theta_{\text{GT}}) \geq D_{\text{Lo}}(\theta)$ and*

$$\Pr[\mathcal{X}_{\text{Hi}}(\theta_{\text{GT}})] (D_{\text{Hi}}(\theta_{\text{GT}}) - D_{\text{Lo}}(\theta_{\text{GT}})) > \Pr[\mathcal{X}_{\text{Hi}}(\theta)] D_{\text{Hi}}(\theta) + D_{\text{KL}}[p(x)||p_\theta(x)];$$

suppose also that (2) that for no such $f_\theta \in \mathcal{F}$ is the MLEO $D_{\text{KL}}[p(x)||p_\theta(x)]$ equal to zero. Then at the global minima (θ^, ϕ^*) of the negative ELBO, the MLEO will be non-zero.*

Proof in Appendix B. Theorem 1 tells us that under conditions (1) and (2) the ELBO can prefer learning likelihood functions f_θ that reconstruct $p(x)$ poorly, even when learning the ground truth likelihood is possible!

Pathology II: The ELBO biases learning of the observation noise variance In practice, the noise variance of the dataset is unknown and it is common to estimate the variance as a hyper-parameter. Here, we show that learning the variance of ϵ either via hyper-parameter search or via direct optimization of the ELBO can be biased.

Theorem 2. *For an observation set of size N , we have that*

$$\underset{\sigma_\epsilon^{(d)^2}}{\text{argmin}} - \text{ELBO}(\theta, \phi, \sigma_\epsilon^{(d)^2}) = \frac{1}{N} \sum_{n=1}^N \mathbb{E}_{q_\phi(z|x_n)} \left[(x_n^{(d)} - f_\theta(z)^{(d)})^2 \right]. \quad (8)$$

Proof in Appendix C. The above theorem shows that the variance σ_ϵ^2 that minimizes the negative ELBO depends on the approximate posterior $q_\phi(z|x)$, and thus even when θ is set to θ_{GT} , the learned σ_ϵ^2 may not equal the ground truth σ_ϵ^2 if $q_\phi(z|x_n)$ is not equal to the true posterior.

In Section 4.2, we provide empirical demonstrations in which training with the ELBO trades-off the quality of the generative model for simple posteriors (Theorem 1), and in which the ELBO selects incorrect values for σ_ϵ^2 (Theorem 2). In Section 6, we describe generally the types of datasets on which we expect the pathologies to occur, and in Section 5, we unfold the consequences of this pathology on unsupervised and semi-supervised tasks.

4.2 Empirical Demonstrations of VAE Training Pathologies

We empirically show that learning σ_ϵ^2 using the ELBO yields biased estimate as predicted in Theorem 2. We also give examples wherein global optima of the VAE training objective correspond to poor generative models. To show that this failure is due to pathologies identified in Theorems 1, we verify that: (A) the learned models have simple posteriors for high mass regions where the ground truth models do not, (B) training with IWAE (complex variational families) results in generally superior generative models and (C) the VAE training cannot be improved meaningfully by methods designed to escape bad local optima, Lagging Inference Networks (LIN) [14]. While examples here are synthetic in order to provide intuition for general failures on down-stream tasks, in Section 6 we describe how each example typifies a class of real datasets on which VAEs can exhibit training pathologies.

Experimental setup We train each model to reach the global optima as follows: we train 10 restarts for each method and hyper-parameter settings – 5 random where we initialize randomly, and 5 random where the decoder and encoder are initialized to ground truth values. We select the restart with the lowest value of the objective function. We fix a sufficiently flexible architecture (one that is

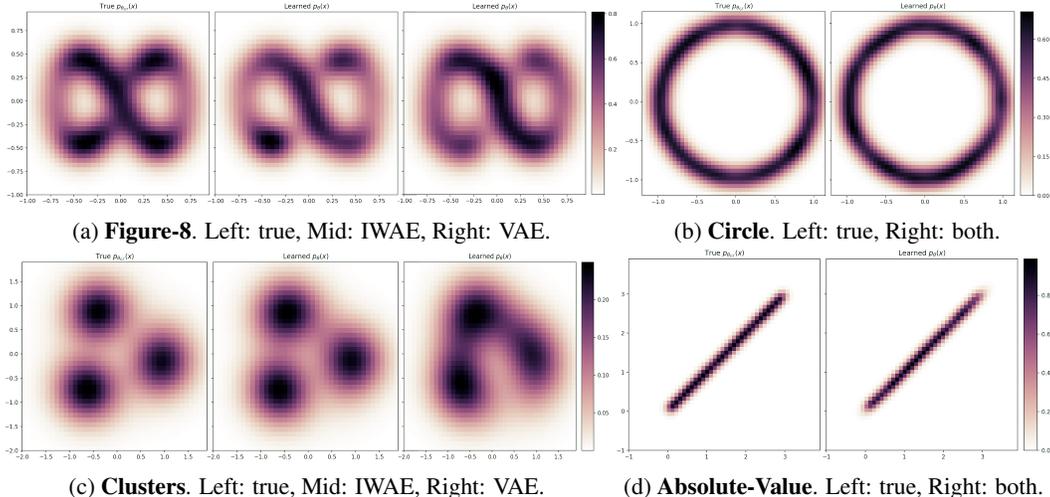


Figure 1: Comparison of true data distributions versus the corresponding learned distributions of VAE and IWAE. When conditions of Theorem 1 are satisfied, in examples (a) and (c), VAE training approximates $p(x)$ poorly and IWAE performs better. When one of the conditions is not met, in examples (b) and (d), then the VAE can learn $p(x)$ as well as IWAE.

significantly more expressive than needed to capture $f_{\theta_{\text{GT}}}$) so that our feasible set \mathcal{F} is diverse enough to include likelihoods with simpler posteriors. Details in Appendix G.

Theorem 2 implies that the ELBO biases noise variance estimates. Consider the ‘‘Spiral Dots’’ Example in Appendix E.5. We perform two experiments. In the first, we fix the noise variance ground-truth ($\sigma_\epsilon^2 = 0.01$), we initialize and train θ, ϕ following the experimental setup above, and finally, we recompute σ_ϵ^2 that maximizes the ELBO for the learned θ, ϕ . In the second experiment, we do the same, but train the ELBO jointly over $\sigma_\epsilon^2, \theta$ and ϕ . Using these two methods of learning the noise, we get 0.014 ± 0.001 and 0.020 ± 0.003 , respectively. The ELBO therefore over-estimates the noise variance by 50% and 100%, respectively.

Approximation of $p(x)$ is poor when Conditions (1) and (2) of Theorem 1 both hold. Consider the ‘‘Figure-8’’ Example visualized in Figure 1a (details in Appendix E.1). Here, the posterior matching objective (PMO) is high for many x ’s, since in the neighborhood of $x \approx 0$ (where $p(x)$ is high), values of z in $[-\infty, -3.0] \cup [3.0, \infty]$ all map to similar values of x . As such, near $x = 0$, the posteriors $p_{\theta_{\text{GT}}}(z|x)$ are multi-modal (Appendix Figure 7d), satisfying condition (1). We verify condition (2) is satisfied by considering all continuous parametrizations of the ‘‘Figure-8’’ curve: any such parametrization will result in a function f_θ for which distant values of z map to similar values near $x = 0$ and thus the PMO will be high. As predicted by Theorem 1, the learned generative model approximates $p(x)$ poorly (Appendix Figure 7a) in order to learn posteriors that are simpler than those of the ground truth model (Appendix Figures 7e vs. 7d).

To show that these issues occur because the MFG variational family over-regularizes the generative model, we compare VAE with LIN and IWAE. As expected, IWAE learns $p(x)$ better than LIN, which outperforms the VAE (Figure 1a and Appendix Table 2). Like the VAE, LIN compromises learning the data distribution in order to learn simpler posteriors since it also uses a MFG variational family (Appendix Figure 8). In contrast, IWAE is able to learn more complex posteriors and thus compromises learning $p(x)$ far less (Appendix Figure 9). However, note that with 20 importance samples, IWAE still does not learn $p(x)$ perfectly. As far as we know, the relationship between the number of importance samples and the complexity of data distributions that can be well approximated by IWAE has not been studied.

We will now demonstrate that both conditions of Theorem 1 are necessary for the VAE training pathology to occur.

Approximation of $p(x)$ may be fine when only condition (2) holds. What happens if the observations with highly non-Gaussian posterior were few in number? For instance, consider the ‘‘Circle’’ Example visualized in Figure 1b (details in Appendix E.2). In this example, the regions of the data-

space that have a non-Gaussian posterior are near $x \approx (1.0, 0.0)$, since $z \in [-\infty, -3.0] \cup [3.0, \infty]$ map to points near $(1.0, 0.0)$. However, since overall number of such points is small, the VAE objective does not need to trade-off capturing $p(x)$ for easy posterior approximation. Indeed, we see that VAE training is capable of recovering $p(x)$, regardless of whether training was initialized randomly or at the ground truth (Appendix Figure 5).

Approximation of $p(x)$ may be fine when only condition (1) holds. We now study the case where the true posterior has a high PMO for a large portion of x 's, but there exist a f_θ in our feasible set \mathcal{F} that approximates $p(x)$ well and has simple posteriors. Consider the ‘‘Absolute-Value’’ Example visualized in Figure 1d (details in Appendix E.3). That the true posteriors are complex (Appendix Figure 6d). However, there is an alternative likelihood $f_\theta(z)$ that explains $p(x)$ equally well and has simpler posteriors (Appendix Figure 6e) and this is the model selected by the VAE objective, regardless of whether training was initialized randomly or at the ground truth.

5 Impact of Pathologies on Downstream Tasks

We demonstrate concrete ways in which VAE training pathologies described in Theorems 1, 2 negatively impact performance on downstream tasks. On unsupervised tasks, we show that IWAE can avoid the negative effects associated with the MFG variational family over-regularizing the generative model, while LIN cannot. However, surprisingly, IWAE cannot outperform the VAE on our semi-supervised tasks as its complex variational family allows the generative model to overfit.

Experiment Setup On unsupervised tasks, we consider only synthetic data, since existing work shows that on real data IWAE learns generative models with higher log data likelihood [24, 7]. For our semi-supervised tasks, we consider both synthetic data as well as 3 UCI data-sets: Diabetic Retinopathy Debreceen [3], Contraceptive Method Choice [1, 11] and the Titanic [1, 45] datasets. In these, we treat the outcome as a partially observed label (observed 10% of the time). These datasets are selected because their classification is hard, and this is the regime in which we expect semi-supervised VAE training to struggle. For the synthetic data sets, we follow the experimental procedure described in Section 4.2. For the UCI data-sets, we split the data 5 different ways into equally sized train/validation/test. On each split of the data, we run 5 random restarts and select the run that yielded the best value on the training objective, computed on the validation set.

To evaluate the quality of the generative model the smooth k NN test statistic [10] on samples from the learned model vs. samples from the training set / ground truth model as an alternative to log-likelihood, since log-likelihood has been shown to be problematic for evaluation [46, 51]. In the semi-supervised case, we also use the smooth k NN test statistic to compare $p(x|y)$ with the learned $p_\theta(x|y)$. Finally, in cases where we may have model mismatch, we also evaluate the mutual information between x and each dimension of the latent space z , using the estimator presented in [28]. For full experimental details, see Appendix G.

5.1 Effects on Unsupervised Downstream Tasks

VAE training pathologies prevent learning disentangled representations In disentangled representation learning, we suppose that each dimension of the latent space corresponds to a task-meaningful concept [39, 5]. Our goal is to infer these meaningful ground truth latent dimensions. It’s been noted in literature that this inference problem is ill-posed - that is, there are an infinite number of likelihood functions (and hence latent codes) that can capture $p(x)$ equally well [29]. Here we show that, more problematically, the VAE objective may *prefer* learning the representations that *entangles* the ground-truth latent dimensions.

Consider data generated by $f_{\theta_{\text{GT}}}(z) = Az + b$. If A is non-diagonal, then the posteriors of this model are correlated Gaussians (poorly approximated by MFGs). Let $A' = AR$, where we define $R = (\Sigma V^\top)^{-1}(\Lambda - \sigma^2 I)^{1/2}$ with an arbitrary diagonal matrix Λ and matrices Σ, V taken from the SVD of A , $A = U\Sigma V^\top$. In this case, $f_\theta = A'z + b$ has the same marginal likelihood as $f_{\theta_{\text{GT}}}$, that is, $p_\theta(x) = p_{\theta_{\text{GT}}}(x) = \mathcal{N}(b, \sigma_\epsilon^2 \cdot I + AA^\top)$. However, since the posteriors of f_θ are uncorrelated, the ELBO will prefer f_θ over $f_{\theta_{\text{GT}}}$! In the latent space corresponding to f_θ , the original *interpretations* of the latent dimensions are now entangled. Similarly, for more complicated likelihood functions, we should expect the ELBO to prefer learning models with simpler posteriors which are not necessarily

ones that are useful for constructing disentangled representations. This bias is reduced in the IWAE training objective.

VAE	Figure-8 Example			Clusters Example		
	$K = 1$ (ground-truth)	$K = 2$	$K = 3$	$K = 1$ (ground-truth)	$K = 2$	$K = 3$
Test $-\text{ELBO}$	-0.127 ± 0.057	-0.260 ± 0.040	-0.234 ± 0.050	4.433 ± 0.049	4.385 ± 0.034	4.377 ± 0.024
Test $\text{avg}_i J(x; z_i)$	2.419 ± 0.027	1.816 ± 0.037	1.296 ± 0.064	1.530 ± 0.011	1.425 ± 0.019	1.077 ± 0.105

IWAE	Figure-8 Example			Clusters Example		
	$K = 1$ (ground-truth)	$K = 2$	$K = 3$	$K = 1$ (ground-truth)	$K = 2$	$K = 3$
Test $-\text{ELBO}$	-0.388 ± 0.044	-0.364 ± 0.051	-0.351 ± 0.045	4.287 ± 0.047	4.298 ± 0.054	4.295 ± 0.049
Test $\text{avg}_i J(x; z_i)$	2.159 ± 0.088	1.910 ± 0.035	1.605 ± 0.087	1.269 ± 0.052	1.321 ± 0.033	1.135 ± 0.110

Table 1: The ELBO prefers learning models with more latent dimensions (and smaller σ_ϵ^2) over the ground truth model ($k = 1$). Although the models preferred by the ELBO have a higher mutual information between the data and learned z 's, the mutual information between dimension of z and the data decreases since with more latent dimensions, the latent space learns ϵ . In contrast, IWAE does not suffer from this pathology. LIN was not included here because it was not able to minimize the negative ELBO as well as the VAE on these data-sets.

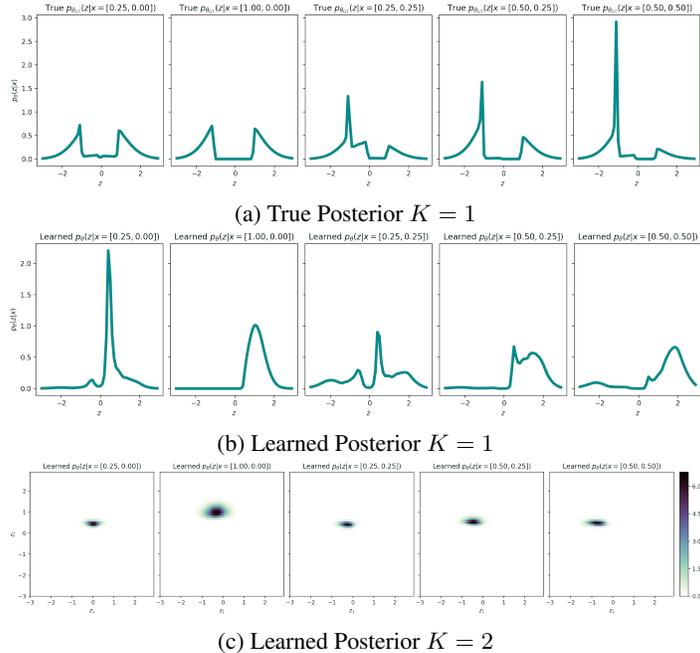


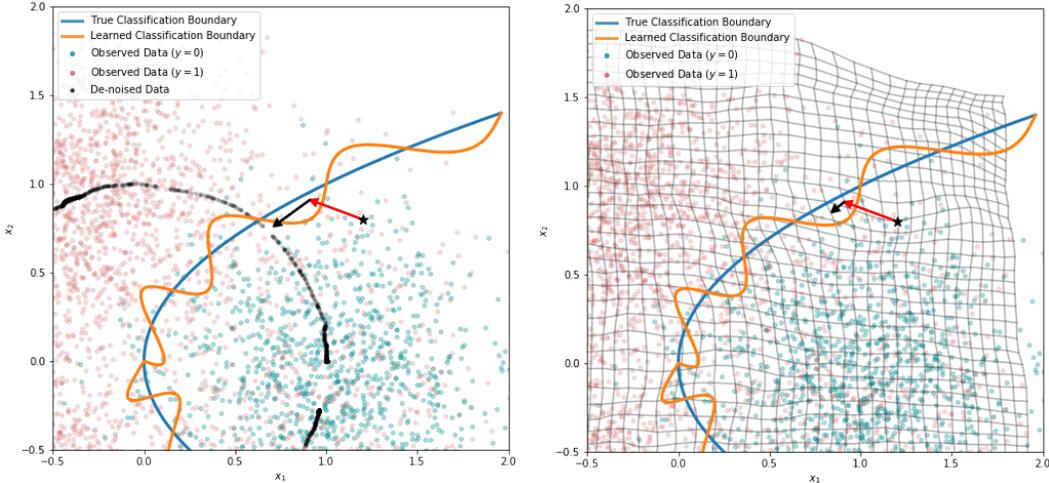
Figure 2: MFG-VAEs learn simpler posteriors as latent dimensionality K increases and as the observation noise σ_ϵ^2 decreases on “Clusters Example” (projected into 5D space). The figure shows that on 5 selected points, the true posterior for the learned model becomes simpler as K increases.

VAE training pathologies prevent learning compressed representations In practice, if the task does not require a specific latent space dimensionality, K , one chooses K that maximizes the $\log p_\theta(x)$. Note that using a higher K and a lower σ_ϵ^2 means we can capture the data distribution with a simpler function $f_\theta(z)$ and hence get simpler posteriors. That is, increasing K alleviates the need to compromise the generative model in order to improve the inference model and leads to better approximation of $p(x)$. Thus, the ELBO will favor model mismatch (K larger than the ground truth) and prevent us from learning highly compressed representations when they are available.

We demonstrate this empirically by embedding the “Figure-8” and “Clusters” Examples into a 5D space using a linear embedding, $A = \begin{pmatrix} 1.0 & 0.0 & 0.5 & 0.2 & -0.8 \\ 0.0 & 1.0 & -0.5 & 0.3 & -0.1 \end{pmatrix}$, and then training a VAE with latent dimensionality $K \in \{1, 2, 3\}$, with $K = 1$ corresponding to the ground-truth model. Training for $K = 1$ is initialized at the ground truth model, and for $K > 2$ we initialize randomly; in each case

we optimize σ_ϵ^2 per-dimension to minimize the negative ELBO. The ELBO prefers models with larger K over the ground truth model ($K = 1$), and that as K increases, the average informativeness of each latent code decreases (see Table 1), since the latent space learns to generate the observation noise ϵ . We confirm that the posteriors become simpler as K increases, lessening the incentive for the VAE to compromise on approximating $p(x)$ (see Figure 2 and Appendix Figure 20). Lastly, we confirm that while LIN also shows preference for higher K 's, IWAE does not (see Table 1).

VAE training pathologies compromises defense against adversarial perturbations We've shown that VAEs prefer increasing the dimensionality of the latent space K and decreasing σ_ϵ^2 . While these models have better approximate $p(x)$, they explain variance due to ϵ using variance due to z , and therefore do not correctly de-noise the data. Furthermore, even when K is fixed at the ground truth, Theorem 2 shows that the ELBO may not be able to identify the correct σ_ϵ^2 . Unfortunately, bias in the noise variance estimate will degrade the performance on tasks requiring correct estimation of the noise. An example of such task is manifold-based defense against adversarial attacks, in which classifiers makes predictions values projected onto the data manifold so as to be robust to adversarial perturbations [16, 34, 42, 15, 18]. In this section we argue that a correct decomposition of the data into $f_\theta(z)$ and ϵ (or "signal" and "noise") is necessary to prevent against certain perturbation-based adversarial attacks. Here, we define an adversarial attack successful if it is able to flip a target classifier's prediction on a given data-point, while its true prediction remains the same (e.g. a picture of a cat is adversarially perturbed so that a classifier predicts "dog", while humans still think the adversarial picture is of a cat).



(a) Projection of adversarial example onto true manifold. (b) Projection of adversarial example onto manifold learned given model mismatch (higher dimensional latent space and smaller observation noise).

Figure 3: Comparison of projection of adversarial example onto ground truth vs. learned manifold. The star represents the original point, perturbed by the red arrow, and then projected onto the manifold by the black arrow.

Assume that our data was generated as follows:

$$\begin{aligned}
 z &\sim p(z) \\
 \epsilon &\sim \mathcal{N}(0, \sigma_\epsilon^2 \cdot I) \\
 x|z &\sim f_{\theta_{\text{GT}}}(z) + \epsilon \\
 y|z &\sim \text{Cat}(g_\psi \circ f_{\theta_{\text{GT}}}(z))
 \end{aligned} \tag{9}$$

Let $\mu_\phi(x)$ denote the mean of encoder and let $M_{\theta, \phi}(x) = f_\theta \circ \mu_\phi(x)$ denote a projection onto the manifold. Our goal is to prevent adversarial attacks on a given discriminative classifier that predicts $y|x$ – that is, we want to ensure that there does not exist any η such that $x_n + \eta$ is classified with a different label than y_n by the learned classifier and not by the ground truth classifier. Since the labels y are computed as a function of the de-noised data, $f_{\theta_{\text{GT}}}(z)$, the true classifier is only defined on the manifold M (marked in blue in Figure 3). As such, any learned classifier (in orange) will

intersect the true classifier on M , but may otherwise diverge from it away from the manifold. This presents a vulnerability against adversarial perturbations, since now any x can be perturbed to cross the learned classifier’s boundary (in orange) to flip its label, while its true label remains the same, as determined by the true classifier (in blue). To protect against this vulnerability, existing methods de-noise the data by projecting it onto the manifold before classifying. Since the true and learned classifiers intersect on the manifold, in order to flip an x ’s label, the x must be perturbed to cross the true classifier’s boundary (and not just the learned classifier’s boundary). This is illustrated in Figure 3a: the black star represents some data point, perturbed (by the red arrow) by an adversary to cross the learned classifier’s boundary but not the true classifier’s boundary. When projected onto the manifold (by the black arrow), the adversarial attack still falls on the same side of the true classifier and the learned classifier, rendering the attack unsuccessful and this method successful.

However, if the manifold is not estimated correctly from the data, this defense may fail. Consider, for example, the case in which $f_\theta(z)$ is modeled with a VAE with a larger dimensional latent space and a smaller observation noise than the ground truth model. Figure 3b shows a uniform grid in x ’s space projected onto the manifold learned by this mismatched model. The figure shows that the learned manifold barely differs from the original space, since the latent space of the VAE compensates for the observation noise ϵ and thus does not de-noise the observation. When the adversarial attack is projected onto the manifold, it barely moves and is thus left as noisy. As the figure shows, the attack crosses the learned classifier’s boundary but not the true boundary and is therefore successful.

5.2 Effects on Semi-Supervised Downstream Tasks

We demonstrate that semi-supervised VAE training suffer from the same pathology identified in Theorem 1: generative model quality is compromised in favor of simple posteriors. Again, as expected, training with complex variational families (i.e. with IWAE) can better capture $p(x)$. However, surprisingly, since the generative model is not regularized by an inflexible variational family, likelihood functions learned by IWAE overfit and, on semi-supervised down-stream tasks like counterfactual data generation, IWAE performs no better than traditional VAE training.

In real datasets, we often have samples from multiple cohorts of the population. General characteristics of the population hold for all cohorts, but each cohort may have different distributions of these characteristics. We formalize this in our model by requiring the cohorts to lie on a shared fixed manifold, while each $p(x|y)$ has a different density on that manifold [27].

In our model, the ground truth posterior $p_{\theta_{\text{GT}}}(z|x) = \int_y p_{\theta_{\text{GT}}}(z, y|x) dy$ will be multi-modal, since for each value of y there are a number of different likely z ’s, each from a different cohort. As such, using a MFG variational family in the semi-supervised objective (Equation 11) will encourage inference to either compromise learning the data-distribution in order to better approximate the posterior, or to learn the data distribution well and poorly approximate the posterior, depending on our prioritization of the two objectives (indicated by our choice of the hyperparameter γ). In the first case, data generation will be compromised but the model will be able to generate realistic counterfactuals. That is, fixing y will allow us to generate realistically from different cohorts $p(x|y)$. In the latter case, the learned model will be able to generate realistic data but not realistic counterfactuals since the model will have collapse the conditional distributions $p_\theta(x|y) \approx p(x)$. That is, $p(x|y)$ will generate identical looking cohort regardless of our choice of y . In short, **VAEs trades-off between generating realistic data and realistic counterfactuals.**

Trade-offs when labels are discrete The trade-off between realistic data and realistic counterfactuals generation is demonstrated in the “Discrete Semi-Circle” Example, visualized in Figure 4 (details in Appendix F.1). the VAE is able to learn the data manifold well (Appendix Figure 14a). However, the learned model has a simple posterior in comparison to the true posterior (Appendix Figure 14f). In fact, the learned $f_\theta(z, y)$ is collapsed to the same function for all values of y (Figure 4c). As a result, $p_\theta(x|y) \approx p_\theta(x)$ under the learned model (Figure 4i vs. 4f). As expected, the same phenomenon occurs when training with LIN (Appendix Figure 15). In contrast, IWAE is able to learn two distinct data conditionals $p_\theta(x|y)$, but it does so at a cost. **Since the IWAE does not regularize the generative model, this leads to overfitting** (Figure 4b). Appendix Table 3 shows that IWAE learns $p(x)$ considerably worse than the VAE, while Appendix Table 4 shows that it learns the $p(x|y)$ significantly better. We see the same pattern in the real data-sets (see Appendix Tables 5 and 6).

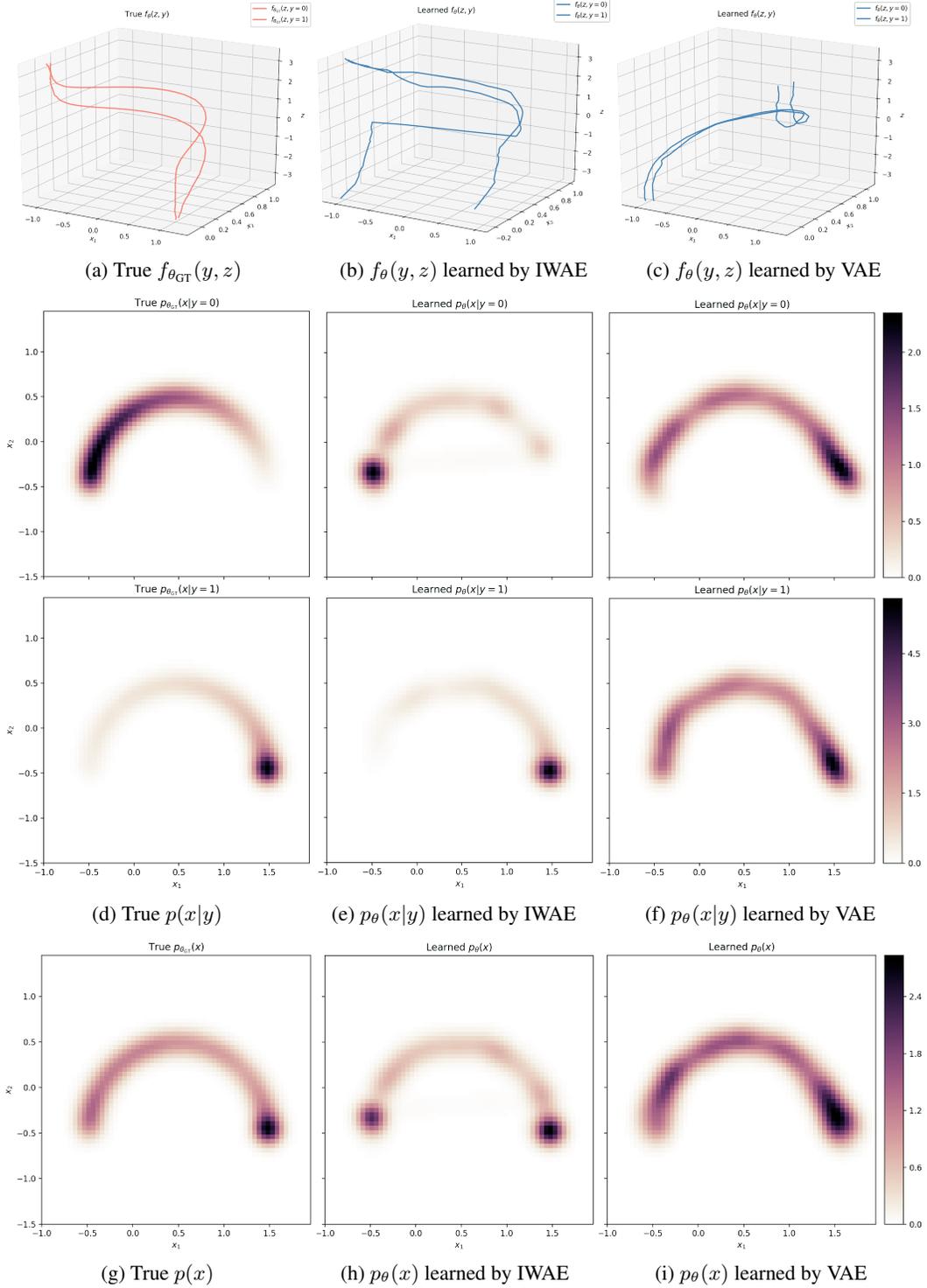


Figure 4: **Discrete Semi-Circle**. Comparison of VAE and IWAE on a semi-supervised example (left column: true, middle column: IWAE, right-column: VAE). The ground truth likelihood function $f_{\theta_{\text{GT}}}(z)$ shows two distinct functions, one for each $y = 0, 1$. The VAE's $f_{\theta}(z)$ is over-regularized by an inflexible variational family and learns two nearly identical functions. The IWAE's $f_{\theta}(z)$ function is un-regularized and learns two distinct but overfitted functions. As a result, both the VAE and IWAE fail to learn $p(x)$ and $p(x|y)$. The VAE learns $p(x)$ better while IWAE learns $p(x|y)$ better.

Trade-offs when labels are continuous When y is discrete, we can lower-bound the number of modes of $p_\theta(z|x)$ by the number of distinct values of y , and choose a variational family that is sufficiently expressive. But when y is continuous, we cannot easily bound the complexity of $p_\theta(z|x)$. In this case, we show that the same trade-off between realistic data and realistic counterfactuals exists, and that there is an *additional* pathology introduced by the discriminator $q_\phi(y|x)$ (Equation 6). Consider the “Continuous Semi-Circle” Example, visualized in Appendix Figure 17b (details in Appendix F.2). Here, since the posterior $p_\theta(y|x)$ is bimodal, encouraging the MFG discriminator $q_\phi(y|x)$ to be predictive will collapse $f_\theta(y, z)$ to the same function for all y (Appendix Figure 17b) So as we increase α (the priority placed on prediction), our predictive accuracy increases at the cost of collapsing $p_\theta(x|y)$ towards $p_\theta(x)$. The latter will result in low quality counterfactuals (see Figure 17c). Like in the discrete case, γ still controls the tradeoff between realistic data and realistic counterfactuals; in the continuous case, α *additionally* controls the tradeoff between realistic counterfactuals and predictive accuracy. Table 4 shows that IWAE is able to learn $p(x)$ better than VAE and LIN, as expected, but *the naive addition of the discriminator to IWAE means that it learns $p(x|y)$ no better than the other two models.*

6 Discussion

VAE training pathologies negatively impact down stream tasks. In Section 5.1 we showed that due to the training pathology we identified in Theorem 1, VAEs may struggle with approximating $p(x)$, learning disentangled representations and learning compressed representations. While as a consequence of Theorem 2, VAEs may struggle with tasks requiring de-noising and manifold learning. Again as a consequence of Theorem 1, in semi-supervised settings, VAEs trade-off generating realistic data with generating realistic counterfactual data. Moreover, we demonstrated that these problems occur at global optima of the VAE training objective. While we showed that on unsupervised tasks these issues are mitigated when we train with IWAE (i.e. increase the complexity of the variational family), the under-regularized generative models of IWAE can overfit and perform no better than VAEs on semi-supervised tasks.

Naive application adaptation of IWAE for semi-supervision introduces new pathologies. The variational family implied by the IWAE objective is not the one given by the IWAE decoder q_ϕ [7]. As such, incorporating a discriminator term $q_\phi(y|x)$ into an IWAE semi-supervised objective is non-trivial, since the real approximate variational family used is complex and requires intractable marginalization over z . Although some get around this intractability by working with lower bounds [43] on $q_\phi(y, z|x)$ marginalized over z , the discriminator in these cases are nonetheless different from the variational posterior. This may be an additional factor of the poor performance of IWAE in the semi-supervised setting with continuous y .

VAE training pathologies can happen for many real datasets. Since our empirical demonstration of the pathology noted in Theorem 1 are synthetic, here we describe how the conditions of Theorem 1 can manifest in real datasets. The “Figure-8” Example in Figure 1a generalizes to any data manifold where the Euclidean distance between two points in a high density region on manifold is (A) less than the length of the geodesic connecting and (B) within 2 standard deviation of observation noise. The “Clusters” Example in Figure 1c generalizes to data distributions that have distinct areas of high density connected by areas of low density. On these datasets, the VAE training objective would prefer compromising the quality of the generative model for posteriors that are easy to approximate. As for the pathology noted in Theorem 2, we expect that the ELBO yields biased estimates of the observation noise whenever the learned model approximates $p(x)$ poorly. As such, on these types of data-sets we expect the performance the unsupervised and semi-supervised downstream tasks to suffer. Furthermore, on we expect the performance of semi-supervised downstream tasks to suffer when $y|x$ is difficult to predict.

7 Conclusion

Variational auto-encoders are widely use by practitioners due to the ease of their implementation and simplicity of their training. When the data consists of images or text, rather than evaluating the model based on metrics alone, we often rely on “gut checks” to make sure that the quality of the latent representations the model learns and the synthetic data (as well as counterfactual data) generated by the model is high (e.g. by reading generate text or inspecting generated images visually

[5, 26]). However, as VAEs are increasingly being used in application where the data is numeric, e.g. in medical or financial domains [37, 19, 50], these intuitive qualitative checks no longer apply. For example, in many medical applications, the original data features themselves (e.g. biometric reading) are difficult to analyze by human experts in raw form. In these cases, where the application touches human lives and potential model error/pathologies are particularly consequential, we need to have a clear theoretical understanding of the failure modes of our models as well as the potential negative consequences on down-stream tasks. In this work, we advocate for exactly this careful approach. We take some first steps towards understanding the precise set of circumstances under which VAE training will result in undesirable models and we pin-point some concrete ways these pathologies can impact important VAE downstream tasks. While we might expect that methods designed to mitigate VAE training pathologies, like IWAE, will also alleviate the negative downstream effects, we find that this is not always so. Our observations point to reasons for further studying the performance VAE alternatives in these applications. Lastly, in this work, we have proposed a set of toy example and dataset characteristics for which VAE training exhibit pathologies. We hope that these examples, along with others proposed in literature, contribute to the definition of a benchmarking dataset of “edge-cases” to test future local latent variable models and inference procedures.

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A The Semi-Supervised VAE Training Objective

We extend VAE model and inference to incorporate partial labels, allowing for some supervision of the latent space dimensions. For this, we use the semi-supervised model first introduced in [23] as the ‘‘M2 model’’. We assume the following generative process:

$$z \sim \mathcal{N}(0, I), \quad \epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2 \cdot I), \quad y \sim p(y), \quad x|y, z = f_\theta(y, z) + \epsilon \quad (10)$$

where y is observed only a portion of the time. Inference objective for this model can be written as a sum of two objectives, a lower bound for the likelihood of M labeled observations and a lower bound for the likelihood for N unlabeled observations:

$$\mathcal{J}(\theta, \phi) = \sum_{n=1}^N \mathcal{U}(x_n; \theta, \phi) + \gamma \cdot \sum_{m=1}^M \mathcal{L}(x_m, y_m; \theta, \phi) \quad (11)$$

where \mathcal{U} and \mathcal{L} lower bound $p_\theta(x)$ and $p_\theta(x, y)$, respectively:

$$\log p_\theta(x, y) \geq \underbrace{\mathbb{E}_{q_\phi(z|x, y)} [-\log p_\theta(x|y, z)] - \log p(y) + D_{\text{KL}}[q_\phi(z|x, y)||p(z)]}_{\mathcal{L}(x, y; \theta, \phi)} \quad (12)$$

$$\log p_\theta(x) \geq \underbrace{\mathbb{E}_{q_\phi(y|x)q_\phi(z|x)} [-\log p_\theta(x|y, z)] + D_{\text{KL}}[q_\phi(y|x)||p(y)] + D_{\text{KL}}[q_\phi(z|x)||p(z)]}_{\mathcal{U}(x; \theta, \phi)} \quad (13)$$

and γ controls their relative weight (as done in [44]). When using IWAE, we substitute the IWAE lower bounds for \mathcal{U} and \mathcal{L} as follows:

$$\log p_\theta(x, y) \geq \underbrace{\mathbb{E}_{z_1, \dots, z_S \sim q_\phi(z|x, y)} \left[\log \frac{1}{S} \frac{p_\theta(x, y, z_s)}{q_\phi(z_s|x, y)} \right]}_{\mathcal{L}(x, y; \theta, \phi)} \quad (14)$$

$$\log p_\theta(x) \geq \underbrace{\mathbb{E}_{(y_1, z_1), \dots, (y_S, z_S) \sim q_\phi(y|x)q_\phi(z|x)} \left[\log \frac{1}{S} \sum_{s=1}^S \frac{p_\theta(x, y_s, z_s)}{q_\phi(y_s|x)q_\phi(z_s|x)} \right]}_{\mathcal{U}(x; \theta, \phi)} \quad (15)$$

B Proof of Theorem 1

Recall the decomposition the negative ELBO in Main Paper Equation 3. In the following discussion, we always set ϕ to be optimal for our choice of θ . Assuming that $p(x)$ is continuous, then for any $\eta \in \mathbb{R}$, we can further decompose the PMO:

$$\begin{aligned} \mathbb{E}_{p(x)} [D_{\text{KL}}[q_\phi(z|x)||p_\theta(z|x)]] &= \Pr[\mathcal{X}_{\text{Lo}}(\theta)] \mathbb{E}_{p(x)|\mathcal{X}_{\text{Lo}}} [D_{\text{KL}}[q_\phi(z|x)||p_\theta(z|x)]] \\ &\quad + \Pr[\mathcal{X}_{\text{Hi}}(\theta)] \mathbb{E}_{p(x)|\mathcal{X}_{\text{Hi}}} [D_{\text{KL}}[q_\phi(z|x)||p_\theta(z|x)]] \end{aligned} \quad (16)$$

where $D_{\text{KL}}[q_\phi(z|x)||p_\theta(z|x)] \leq \eta$ on $\mathcal{X}_{\text{Lo}}(\theta)$, $D_{\text{KL}}[q_\phi(z|x)||p_\theta(z|x)] > \eta$ on $\mathcal{X}_{\text{Hi}}(\theta)$, with $\mathcal{X}_i(\theta) \subseteq \mathcal{X}$; where $\mathbb{E}_{p(x)|\mathcal{X}_i}$ is the expectation over $p(x)$ restricted to $\mathcal{X}_i(\theta)$ and renormalized, and $\Pr[\mathcal{X}_i]$ is the probability of $\mathcal{X}_i(\theta)$ under $p(x)$. Let us denote the expectation in first term on the right hand side of Equation 16 as $D_{\text{Lo}}(\theta)$ and the expectation in the second term as $D_{\text{Hi}}(\theta)$.

Let $f_{\theta_{\text{GT}}} \in \mathcal{F}$ be the ground truth likelihood function, for which we may assume that the MLE objective (MLEO) term is zero. We can now state our claim:

Theorem. *Suppose that there exist an $\eta \in \mathbb{R}$ such that $\Pr[\mathcal{X}_{\text{Hi}}(\theta_{\text{GT}})] D_{\text{Hi}}(\theta_{\text{GT}})$ is greater than $\Pr[\mathcal{X}_{\text{Lo}}(\theta_{\text{GT}})] D_{\text{Lo}}(\theta_{\text{GT}})$. Suppose that (1) there exist an $f_\theta \in \mathcal{F}$ such that $D_{\text{Lo}}(\theta_{\text{GT}}) \geq D_{\text{Lo}}(\theta)$ and*

$$\Pr[\mathcal{X}_{\text{Hi}}(\theta_{\text{GT}})] (D_{\text{Hi}}(\theta_{\text{GT}}) - D_{\text{Lo}}(\theta_{\text{GT}})) > \Pr[\mathcal{X}_{\text{Hi}}(\theta)] D_{\text{Hi}}(\theta) + D_{\text{KL}}[p(x)||p_\theta(x)];$$

suppose also that (2) that for no such $f_\theta \in \mathcal{F}$ is the MLEO $D_{\text{KL}}[p(x)||p_\theta(x)]$ equal to zero. Then at the global minima (θ^, ϕ^*) of the negative ELBO, the MLEO will be non-zero.*

Proof. The proof is straightforward. Condition (1) of the theorem implies that the negative ELBO of f_θ will be lower than that of $f_{\theta_{\text{GT}}}$. That is, we can write:

$$-\text{ELBO}(\theta_{\text{GT}}, \phi_{\text{GT}}) = \Pr[\mathcal{X}_{\text{Hi}}(\theta_{\text{GT}})] D_{\text{Hi}}(\theta_{\text{GT}}) + \Pr[\mathcal{X}_{\text{Lo}}(\theta_{\text{GT}})] D_{\text{Lo}}(\theta_{\text{GT}}) \quad (17)$$

$$= \Pr[\mathcal{X}_{\text{Hi}}(\theta_{\text{GT}})] D_{\text{Hi}}(\theta_{\text{GT}}) + (1 - \Pr[\mathcal{X}_{\text{Hi}}(\theta_{\text{GT}})]) D_{\text{Lo}}(\theta_{\text{GT}}) \quad (18)$$

$$= \Pr[\mathcal{X}_{\text{Hi}}(\theta_{\text{GT}})] (D_{\text{Hi}}(\theta_{\text{GT}}) - D_{\text{Lo}}(\theta_{\text{GT}})) + D_{\text{Lo}}(\theta_{\text{GT}}) \quad (19)$$

$$> \underbrace{\Pr[\mathcal{X}_{\text{Hi}}(\theta)] D_{\text{Hi}}(\theta) + \Pr[\mathcal{X}_{\text{Lo}}(\theta)] D_{\text{Lo}}(\theta) + D_{\text{KL}}[p(x)||p_\theta(x)]}_{-\text{ELBO}(\theta, \phi)} \quad (20)$$

So we have that $-\text{ELBO}(\theta_{\text{GT}}, \phi_{\text{GT}}) > -\text{ELBO}(\theta, \phi)$. Note again that by construction ϕ_{GT} and ϕ are both optimal for θ_{GT} and θ , respectively.

Furthermore, if there is an $f_{\theta'} \in \mathcal{F}$ such that $-\text{ELBO}(\theta', \phi') < -\text{ELBO}(\theta, \phi)$, then it must also satisfy the conditions in assumption (1) and, hence, the global minima of the negative ELBO satisfy the conditions in assumption (1). By assumption (2), at the global minima of the negative ELBO, the MLEO $D_{\text{KL}}[p(x)||p_\theta(x)]$ cannot be equal to zero. \square

C Proof of Theorem 2

In practice, the noise variance of the dataset is unknown and it is common to estimate the variance as a hyper-parameter. Here, we show that learning the variance of ϵ either via hyper-parameter search or via direct optimization of the ELBO can be biased.

Theorem. For an observation set of size N , we have that

$$\underset{\sigma_\epsilon^{(d)^2}}{\operatorname{argmin}} -\text{ELBO}(\theta, \phi, \sigma_\epsilon^{(d)^2}) = \frac{1}{N} \sum_{n=1}^N \mathbb{E}_{q_\phi(z|x_n)} \left[(x_n^{(d)} - f_\theta(z)^{(d)})^2 \right]. \quad (21)$$

Proof. We rewrite the negative ELBO:

$$\underset{\sigma_\epsilon^{(d)^2}}{\operatorname{argmin}} -\text{ELBO}(\theta, \phi, \sigma_\epsilon^2) \quad (22)$$

$$= \underset{\sigma_\epsilon^{(d)^2}}{\operatorname{argmin}} \mathbb{E}_{p(x)} \left[\mathbb{E}_{q_\phi(z|x)} [-\log p_\theta(x|z)] + D_{\text{KL}}[q_\phi(z|x)||p(z)] \right] \quad (23)$$

$$= \underset{\sigma_\epsilon^{(d)^2}}{\operatorname{argmin}} \mathbb{E}_{p(x)} \left[\mathbb{E}_{q_\phi(z|x)} [-\log p_\theta(x|z)] \right] \quad (24)$$

$$= \underset{\sigma_\epsilon^{(d)^2}}{\operatorname{argmin}} \mathbb{E}_{p(x)} \left[\mathbb{E}_{q_\phi(z|x)} \left[-\sum_{d=1}^D \log \left(\frac{1}{\sqrt{2\pi\sigma_\epsilon^{(d)^2}}} \cdot \exp \left(\frac{-(x^{(d)} - f_\theta(z)^{(d)})^2}{2\sigma_\epsilon^{(d)^2}} \right) \right) \right] \right] \quad (25)$$

$$= \underset{\sigma_\epsilon^{(d)^2}}{\operatorname{argmin}} \sum_{d=1}^D \mathbb{E}_{p(x)} \left[\mathbb{E}_{q_\phi(z|x)} \left[\log \left(\sqrt{2\pi\sigma_\epsilon^{(d)^2}} \right) + \frac{(x^{(d)} - f_\theta(z)^{(d)})^2}{2\sigma_\epsilon^{(d)^2}} \right] \right] \quad (26)$$

$$= \underset{\sigma_\epsilon^{(d)^2}}{\operatorname{argmin}} \sum_{d=1}^D \mathbb{E}_{p(x)} \left[\mathbb{E}_{q_\phi(z|x)} \left[\log \left(\sigma_\epsilon^{(d)} \right) + \frac{(x^{(d)} - f_\theta(z)^{(d)})^2}{2\sigma_\epsilon^{(d)^2}} \right] \right] \quad (27)$$

$$= \underset{\sigma_\epsilon^{(d)^2}}{\operatorname{argmin}} \sum_{d=1}^D \log \left(\sigma_\epsilon^{(d)} \right) + \frac{1}{2\sigma_\epsilon^{(d)^2}} \cdot \underbrace{\mathbb{E}_{p(x)} \left[\mathbb{E}_{q_\phi(z|x)} \left[(x^{(d)} - f_\theta(z)^{(d)})^2 \right] \right]}_{C(\theta, \phi, d)} \quad (28)$$

Setting the gradient of the above with respect to σ_ϵ^2 equal to zero yields the following:

$$0 = -\frac{\partial}{\partial \sigma_\epsilon^{(d)}} \text{ELBO}(\theta, \phi, \sigma_\epsilon^{(d)}) \quad (29)$$

$$= \frac{\sigma_\epsilon^{(d)^2} - C(\theta, \phi, d)}{\sigma_\epsilon^{(d)^3}}. \quad (30)$$

Thus, we can write,

$$\sigma_\epsilon^{(d)2} = C(\theta, \phi, d) = \mathbb{E}_{p(x)} \left[\mathbb{E}_{q_\phi(z|x)} \left[(x^{(d)} - f_\theta(z)^{(d)})^2 \right] \right] \quad (31)$$

$$\approx \frac{1}{N} \sum_{n=1}^N \mathbb{E}_{q_\phi(z|x_n)} \left[(x_n^{(d)} - f_\theta(z)^{(d)})^2 \right] \quad (32)$$

□

D Quantitative Results

In this section we present the quantitative results for the paper.

Approximation of $p(x)$ is poor when Conditions (1) and (2) of Theorem 1 both hold. Table 2 shows that on synthetic data-sets for which Theorem 1 hold, the VAE objective (even with a better training algorithm, LIN) approximates $p(x)$ poorly, while methods with a more complex variational family (IWAE) do not. Visualization of the posterior (in Appendix H) confirm that the VAE objective underfits the generative model in order to learn a simpler posterior, whereas the IWAE objective does not: for the “Figure-8 Example”, see Figures 7, 8 and 9 and for the “Clusters Example”, see Figures 10, 11 and 12). In these two examples, we further see the ELBO’s regularizing effect on the learned f_θ . On the “Figure-8 Example”, the learned f_θ ensures that x ’s generated from $z \in [-\infty, -3] \cup [3, \infty]$ are sufficiently different from x ’s generated from $z \approx 0$: $f_\theta(z)$ curls away from the center $z \approx 0$ and thus simplifies the posterior. On the “Clusters Examples”, the learned f_θ has less pronounced changes in slope, and thus a simpler posterior.

VAE training pathologies prevent learning compressed representations Table 1 shows that the VAE objective prefers learning models with more latent dimensions and smaller observation noise σ_ϵ^2 over the ground truth model. As a result, the learned latent dimensions compensate for the observation noise and each become less informative. Specifically it shows that as we increase the dimensionality of the latent space, the negative ELBO decreases and the average mutual information between the latent codes and the data decreases. In contrast, we do not observe the same trend with IWAE because IWAE does not need to compromise learning $p(x)$ as much in order to learn simpler posteriors. Figures 2 and 20 (for “Figure-8 Example” and the “Clusters Example”, respectively) show the posteriors of the learned models become simpler as we increase the dimensionality of the latent space.

VAEs trades-off between generating realistic data and realistic counterfactuals in semi-supervision When using a MFG variational family to approximate $p_\theta(z|x)$ on unlabeled data, and in the continuous case using a MFG variational family to approximate $p_\theta(y|x)$, the function f_θ is over-regularized, causing the learned data condition $p_\theta(x|y)$ to match $p(x)$. This prevents us from generating realistic counterfactuals. Tables 3 and 5 (synthetic and real, respectively) show that while IWAE generally approximates $p(x)$ better than a VAE, both are able to learn $p(x)$ significantly better than they are able to learn $p(x|y)$ (see Tables 4 and 6). Figures 14 and 16 (for the VAE and IWAE, respectively) demonstrate that in the discrete under the learned models, the function $f_\theta(y, z)$ collapses to the same function for all values of y . Figures 17 and 19 show the same for the continuous case.

Data	IWAE	LIN	VAE
Clusters	0.057 ± 0.028	0.347 ± 0.057	0.361 ± 0.083
Fig-8	0.036 ± 0.013	0.040 ± 0.081	0.066 ± 0.014

Table 2: Comparison unsupervised learned vs. true data distributions via the smooth k NN test (lower is better). Hyper-parameters selected via smaller value of the loss function on the validation set.

Data	IWAE	LIN	VAE
Discrete Semi-Circle	0.694 ± 0.096	0.703 ± 0.315	0.196 ± 0.078
Continuous Semi-Circle	0.015 ± 0.011	0.128 ± 0.094	0.024 ± 0.014

Table 3: Comparison of semi-supervised learned vs. true data distributions via the smooth k NN test (lower is better). Hyper-parameters selected via the smooth k NN test-statistic computed on the data marginals.

Data	IWAE		LIN		VAE	
	Cohort 1	Cohort 2	Cohort 1	Cohort 2	Cohort 1	Cohort 2
Discrete Semi-Circle	1.426 ± 1.261	1.698 ± 0.636	18.420 ± 1.220	10.118 ± 0.996	15.206 ± 1.200	11.501 ± 1.300
Continuous Semi-Circle	15.951 ± 3.566	14.416 ± 1.402	15.321 ± 1.507	17.530 ± 1.509	13.128 ± 0.825	16.046 ± 1.019

Table 4: Comparison of semi-supervised learned $p_\theta(x|y)$ with ground truth $p(x|y)$ via the smooth k NN test statistic (smaller is better). Hyper-parameters selected via smallest smooth k NN test statistic computed on the data marginals. For the discrete data, the cohorts are $p(x|y = 0)$ and $p(x|y = 1)$, and for the continuous data, the cohorts are $p(x|y = -3.5)$ and $p(x|y = 3.5)$.

	IWAE	VAE
Diabetic Retinopathy	3.571 ± 2.543	6.206 ± 1.035
Contraceptive	1.740 ± 0.290	2.147 ± 0.225
Titanic	2.794 ± 1.280	1.758 ± 0.193

Table 5: Comparison of semi-supervised learned vs. true data distributions via the smooth k NN test (lower is better). Hyper-parameters selected via the smooth k NN test-statistic computed on the data marginals.

	IWAE			VAE		
	Cohort 1	Cohort 2	Cohort 3	Cohort 1	Cohort 2	Cohort 3
Diabetic Retinopathy	4.240 ± 1.219	4.357 ± 3.417	N/A	5.601 ± 0.843	8.008 ± 1.096	N/A
Contraceptive	7.838 ± 1.138	5.521 ± 3.519	6.626 ± 2.571	5.388 ± 0.788	4.994 ± 0.932	3.722 ± 0.488
Titanic	3.416 ± 0.965	6.923 ± 1.924	N/A	3.730 ± 0.866	8.572 ± 1.766	N/A

Table 6: Comparison of semi-supervised learned vs. true conditional distributions $p(x|y)$ via the smooth k NN test (lower is better). Hyper-parameters selected via the smooth k NN test-statistic computed on the data marginals.

E Unsupervised Pedagogical Examples

In this section we describe in detail the unsupervised pedagogical examples used in the paper and the properties that cause them to trigger the VAE pathologies. For each one of these example decoder functions, we fit a surrogate neural network f_θ using full supervision (ensuring that the $\text{MSE} < 1e-4$ and use that f_θ to generate the actual data used in the experiments.

E.1 Figure-8 Example

Generative Process:

$$\begin{aligned}
 z &\sim \mathcal{N}(0, 1) \\
 \epsilon &\sim \mathcal{N}(0, \sigma_\epsilon^2 \cdot I) \\
 u(z) &= (0.6 + 1.8 \cdot \Phi(z)) \pi \\
 x|z &= \underbrace{\begin{bmatrix} \frac{\sqrt{2}}{2} \cdot \frac{\cos(u(z))}{\sin(u(z))^2+1} \\ \sqrt{2} \cdot \frac{\cos(u(z)) \sin(u(z))}{\sin(u(z))^2+1} \end{bmatrix}}_{f_{\theta_{\text{GT}}}(z)} + \epsilon
 \end{aligned} \tag{33}$$

where $\Phi(z)$ is the Gaussian CDF and $\sigma_\epsilon^2 = 0.02$ (see Figure 7).

Properties: In this example, values of z on $[-\infty, -3.0]$, $[3.0, \infty]$ and in small neighborhoods of $z = 0$ all produce similar values of x , namely $x \approx 0$; as such, the true posterior $p_{\theta_{\text{GT}}}(z|x)$ is multi-modal in the neighborhood of $x = 0$ (see Figure 7d), leading to high PMO. Additionally, in the neighborhood of $x \approx 0$, $p(x)$ is high. Thus, condition (1) of Theorem 1 is satisfied. One can verify condition (2) is satisfied by considering all continuous parametrizations of a figure-8 curve. Any such parametrization will result in a f_θ for which far-away values of z lead to nearby values of x and thus in high PMO value for points near $x = 0$.

E.2 Circle Example

Generative Process:

$$\begin{aligned}
 z &\sim \mathcal{N}(0, 1) \\
 \epsilon &\sim \mathcal{N}(0, \sigma_\epsilon^2 \cdot I) \\
 x|z &= \underbrace{\begin{bmatrix} \cos(2\pi \cdot \Phi(z)) \\ \sin(2\pi \cdot \Phi(z)) \end{bmatrix}}_{f_{\theta_{\text{GT}}}(z)} + \epsilon
 \end{aligned} \tag{34}$$

where $\Phi(z)$ is the Gaussian CDF and $\sigma_\epsilon^2 = 0.01$ (see Figure 5).

Properties: In this example, the regions of the data-space that have a non-Gaussian posterior are near $x \approx [1.0, 0.0]$, since in that neighborhood, $z \in [-\infty, -3.0]$ and $z \in [3.0, \infty]$ both generate nearby values of x . Thus, this model only satisfies condition 2 of Theorem 1. However, since overall the number of x 's for which the posterior is non-Gaussian are few, the VAE objective does not need to trade-off capturing $p(x)$ for easy posterior approximation. We see that traditional training is capable of recovering $p(x)$, regardless of whether training was initialized randomly or at the ground truth (see Figure 5).

E.3 Absolute-Value Example

Generative Process:

$$\begin{aligned}
 z &\sim \mathcal{N}(0, 1) \\
 \epsilon &\sim \mathcal{N}(0, \sigma_\epsilon^2 \cdot I) \\
 x|z &= \underbrace{\begin{bmatrix} |\Phi(z)| \\ \Phi(z) \end{bmatrix}}_{f_{\theta_{\text{GT}}}(z)} + \epsilon
 \end{aligned} \tag{35}$$

where $\Phi(z)$ is the Gaussian CDF and $\sigma_\epsilon^2 = 0.01$ (see Figure 6).

Properties: In this example, the posterior under $f_{\theta_{\text{GT}}}$ cannot be well approximated using a MFG variational family (see Figure 6d). However, there does exist an alternative likelihood function $f_\theta(z)$ (see 6b) that explains $p(x)$ equally well and has simpler posterior 6e. As such, this model only satisfies condition 1 of Theorem 1.

E.4 Clusters Example

Generative Process:

$$\begin{aligned}
z &\sim \mathcal{N}(0, 1) \\
\epsilon &\sim \mathcal{N}(0, \sigma_\epsilon^2 \cdot I) \\
u(z) &= \frac{2\pi}{1 + e^{-\frac{1}{2}\pi z}} \\
t(u) &= 2 \cdot \tanh(10 \cdot u - 20 \cdot \lfloor u/2 \rfloor - 10) + 4 \cdot \lfloor u/2 \rfloor + 2 \\
x|z &= \underbrace{\begin{bmatrix} \cos(t(u(z))) \\ \sin(t(u(z))) \end{bmatrix}}_{f_{\theta_{\text{GT}}}(z)} + \epsilon
\end{aligned} \tag{36}$$

where $\sigma_\epsilon^2 = 0.2$.

Properties: In this example, $f_{\theta_{\text{GT}}}$ a step function embedded on a circle. Regions in which $\frac{df_{\theta_{\text{GT}}}^{-1}}{dx}$ is high (i.e. the steps) correspond to regions in which $p(x)$ is high. The interleaving of high density and low density regions on the manifold yield a multi-modal posterior (see Figure 10d). For this model, both conditions of Theorem 1 hold. In this example, we again see that the VAE objective learns a model with a simpler posterior (see Figure 10e) at the cost of approximating $p(x)$ well (see Figure 10a).

E.5 Spiral Dots Example

Generative Model:

$$\begin{aligned}
z &\sim \mathcal{N}(0, 1) \\
\epsilon &\sim \mathcal{N}(0, \sigma_\epsilon^2 \cdot I) \\
u(z) &= \frac{4\pi}{1 + e^{-\frac{1}{2}\pi z}} \\
t(u) &= \tanh(10 \cdot u - 20 \cdot \lfloor u/2 \rfloor - 10) + 2 \cdot \lfloor u/2 \rfloor + 1 \\
x|z &= \underbrace{\begin{bmatrix} t(u(z)) \cdot \cos(t(u(z))) \\ t(u(z)) \cdot \sin(t(u(z))) \end{bmatrix}}_{f_{\theta_{\text{GT}}}(z)} + \epsilon
\end{aligned} \tag{37}$$

where $\sigma_\epsilon^2 = 0.01$.

Properties: In this example, $f_{\theta_{\text{GT}}}$ a step function embedded on a spiral. Regions in which $\frac{df_{\theta_{\text{GT}}}^{-1}}{dx}$ is high (i.e. the steps) correspond to regions in which $p(x)$ is high. The interleaving of high density and low density regions on the manifold yield a multi-modal posterior (see Figure 13d). In this example, we again see that the VAE objective learns a model with a simpler posterior (see Figure 13e) at the cost of approximating $p(x)$ well (see Figure 13a). Furthermore, for this model the VAE objective highly misestimates the observation noise.

F Semi-Supervised Pedagogical Examples

In this section we describe in detail the semi-supervised pedagogical examples used in the paper and the properties that cause them to trigger the VAE pathologies. For each one of these example

decoder functions, we fit a surrogate neural network f_θ using full supervision (ensuring that the $\text{MSE} < 1e - 4$ and use that f_θ to generate the actual data used in the experiments.

F.1 Discrete Semi-Circle Example

Generative Process:

$$\begin{aligned}
 z &\sim \mathcal{N}(0, 1) \\
 y &\sim \text{Bern}\left(\frac{1}{2}\right) \\
 \epsilon &\sim \mathcal{N}(0, \sigma_\epsilon^2 \cdot I) \\
 x|y, z &= \underbrace{\begin{bmatrix} \cos\left(\mathbb{I}(y=0) \cdot \pi \cdot \sqrt{\Phi(z)} + \mathbb{I}(y=1) \cdot \pi \cdot \Phi(z)^3\right) \\ \sin\left(\mathbb{I}(y=0) \cdot \pi \cdot \sqrt{\Phi(z)} + \mathbb{I}(y=1) \cdot \pi \cdot \Phi(z)^3\right) \end{bmatrix}}_{f_{\theta_{\text{GT}}}(y, z)} + \epsilon
 \end{aligned} \tag{38}$$

where Φ is the CDF of a standard normal and $\sigma_\epsilon^2 = 0.01$.

Properties: We designed this data-set to specifically showcase issues with the semi-supervised VAE objective. As such, we made sure that the data marginal $p(x)$ of this example will be learned well using unsupervised VAE (trained on the x 's only) This way we can focus on the new issues introduced by the semi-supervised objective.

For this ground-truth model, the posterior of the un-labeled data $p_{\theta_{\text{GT}}}(z|x)$ is bimodal, since there are two functions that could have generated each x : $f_{\theta_{\text{GT}}}(y=0, z)$ and $f_{\theta_{\text{GT}}}(y=1, z)$. As such, approximating this posterior with a MFG will encourage the semi-supervised objective to find a model for which $f_{\theta_{\text{GT}}}(y=0, z) = f_{\theta_{\text{GT}}}(y=1, z)$ (see Figure 14b). When both functions collapse to the same function, $p_\theta(x|y) \approx p_\theta(x)$ (see Figure 14c). This will prevent the learned model from generating realistic counterfactuals.

F.2 Continuous Semi-Circle Example

Generative Process:

$$\begin{aligned}
 z &\sim \mathcal{N}(0, 1) \\
 y &\sim \mathcal{N}(0, 1) \\
 h(y) &= B^{-1}(\Phi(y); 0.2, 0.2) \\
 \epsilon &\sim \mathcal{N}(0, \sigma_\epsilon^2 \cdot I) \\
 x|y, z &= \underbrace{\begin{bmatrix} \cos\left(h(y) \cdot \pi \cdot \sqrt{\Phi(z)} + (1 - h(y)) \cdot \pi \cdot \Phi(z)^3\right) \\ \sin\left(h(y) \cdot \pi \cdot \sqrt{\Phi(z)} + (1 - h(y)) \cdot \pi \cdot \Phi(z)^3\right) \end{bmatrix}}_{f_{\theta_{\text{GT}}}(y, z)} + \epsilon
 \end{aligned} \tag{39}$$

where Φ is the CDF of a standard normal and $B^{-1}(\cdot; \alpha, \beta)$ is the inverse CDF of the beta distribution.

Properties: As in the ‘‘Discrete Semi-Circle Example’’, we designed this data-set to have a $p(x)$ that the VAE objective would learn well so we can focus on the new issues introduced by the semi-supervised objective. The dataset demonstrates the same pathologies in the semi-supervised objective as shown by ‘‘Discrete Semi-Circle Example’’ with the addition of yet another pathology: when since the posterior $p_\theta(y|x)$ is bimodal in this example, encouraging a MFG $q_\phi(y|x)$ discriminator to be predictive will collapse $f_\theta(y, z)$ to the same function for all values of y (see Figure 17b) As such, as we increase α , the better our predictive accuracy will be but the more $p_\theta(x|y) \rightarrow p_\theta(x)$, causing the learned model to generate poor quality counterfactuals (see Figure 17c).

G Experimental Details

Initialization at Global Optima of the VAE Objective The decoder function f_θ is initialized to the ground-truth using full supervision given the ground-truth z 's and $f_{\theta_{\text{GT}}}$. The encoder is initialized to ϕ_{GT} by fixing the decoder at the ground-truth and maximizing the ELBO (with the 10 random restarts). We fix the observation error σ_ϵ^2 to that of the ground truth model, and we fix a sufficiently flexible architecture – one that is significantly more expressive than needed to capture $f_{\theta_{\text{GT}}}$ – to ensure that, if there exists a f_θ with simpler posteriors, it would be included in our feasible set \mathcal{F} . Lastly, we select the restart that yields the lowest value of the objective function.

Synthetic Datasets We use 4 synthetic data-sets for unsupervised VAEs (described in Appendix E), and 2 synthetic data-sets for semi-supervised VAEs (described in Appendix F), and generate 5 versions of each data-set (each with 5000/2000/2000 train/validation/test points). We use 3 real semi-supervised data-sets: Diabetic Retinopathy Debrecen [3], Contraceptive Method Choice [1, 11] and the Titanic [1, 45] datasets, each with 10% observed labels, split in 5 different ways equally into train/validation/test.

Architectures On the synthetic data-sets, we use a leaky-ReLU encoder/decoder with 3 hidden layers, each 50 nodes. On the UCI data-sets, we use a leaky-ReLU encoder/decoder with 3 hidden layers, each 100 nodes.

Optimization For optimization, we use the Adam optimizer [21] with a learning rate of 0.001 and a mini-batch size of 100. We train for 100 epochs on synthetic data and for 20000 on real data (and verified convergence). We trained 5 random restarts on each of the split of the data. For semi-supervised data-sets with discrete labels, we used continuous relaxations of the categorical distribution with temperature 2.2 [17] as the variational family in order to use the reparameterization trick [25].

Baselines For our baselines, we compare the performance of a MFG-VAE with that of a VAE trained with the Lagging Inference Networks (LIN) algorithm (still with a MFG variational family), since the algorithm claims to be able to escape local optima in training. Since the pathologies we describe are global optima, we do not expect LIN to mitigate the issues. We use Importance Weighted Autoencoders (IWAE) as an example of a inference algorithm that uses a more complex variational family. Since the pathologies described are exacerbated by a limited variational family, we expect IWAE to out-perform the other two approaches. For each method, we select the hyper-parameters for which the best restart yields the best log-likelihood (using the smooth k NN test-statistic, described below).

Hyper-parameters When using IWAE, let S be the number of importance samples used. When using the Lagging Inference Networks, let T be the threshold for determining whether the inference network objective has converged, and let R be the number of training iterations for which the loss is averaged before comparing with the threshold. When using semi-supervision, α determines the weight of the discriminator, and γ determines the weight of the labeled objective, \mathcal{L} . We grid-searched over all combination of the following sets of parameters:

Unsupervised datasets:

- IWAE: $S \in \{3, 10, 20\}$
- Lagging Inference Networks: $T \in \{0.05, 0.1\}$, $R \in \{5, 10\}$

Semi-supervised synthetic datasets:

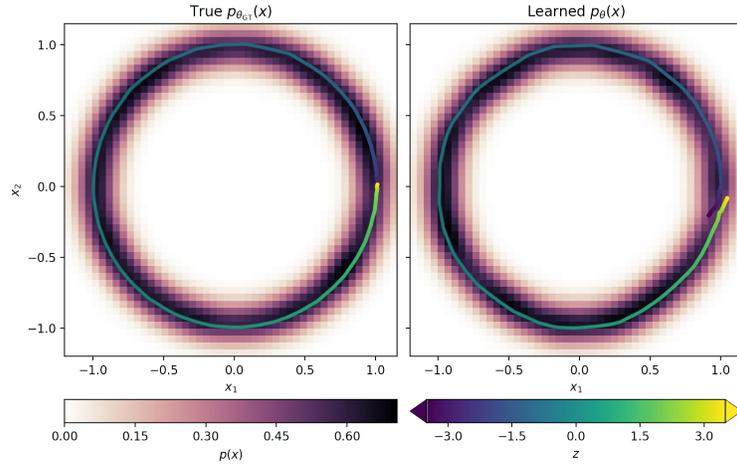
- IWAE: $S \in \{3, 10, 20\}$
- Lagging Inference Networks: $T \in \{0.05, 0.1\}$, $R \in \{5, 10\}$
- All methods: $\alpha \in \{0.0, 0.1, 1.0\}$, $\gamma \in \{0.5, 1.0, 2.0, 5.0\}$

Semi-supervised real datasets:

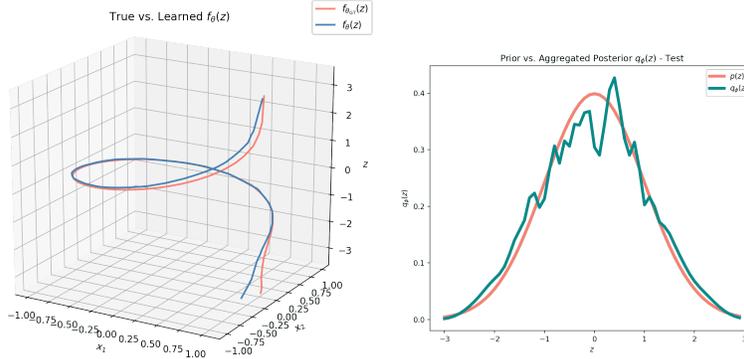
- IWAE: $S \in \{3, 10, 20\}$
- Lagging Inference Networks: $T \in \{0.05, 0.1\}$, $R \in \{5, 10\}$
- All methods: $\alpha \in \{0.0, 0.1, 1.0\}$, $\gamma \in \{0.5, 1.0, 2.0, 5.0\}$, $\sigma_\epsilon^2 \in \{0.01, 0.5\}$. On Titanic dimensionality of z is $\in \{1, 2\}$, on Contraceptive and Diabetic Retinopathy $\in \{2, 5\}$.

Hyper-parameters Selection For each method, we selected the hyper-parameters that yielded the smallest value of the smooth k NN test statistic (indicating that they learned the $p(x)$ best).

H Qualitative examples to support necessity of both conditions of Theorem 1

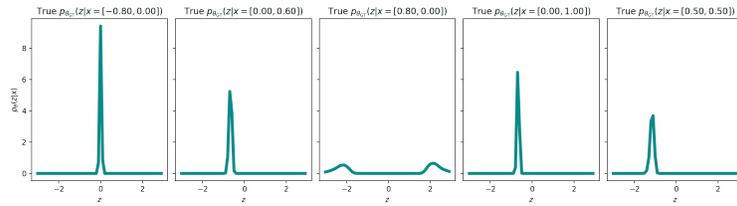


(a) True vs. learned $p_{\theta}(x)$, and learned vs. true $f_{\theta}(z)$, colored by the value of z .

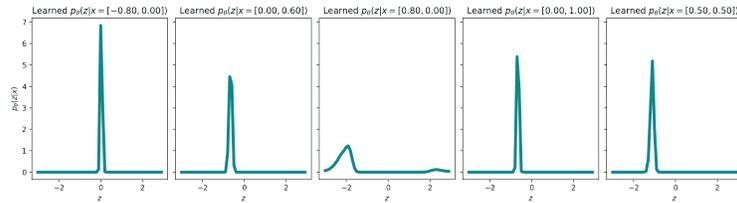


(b) True vs. learned $f_{\theta}(x)$

(c) Aggregated posterior vs. prior

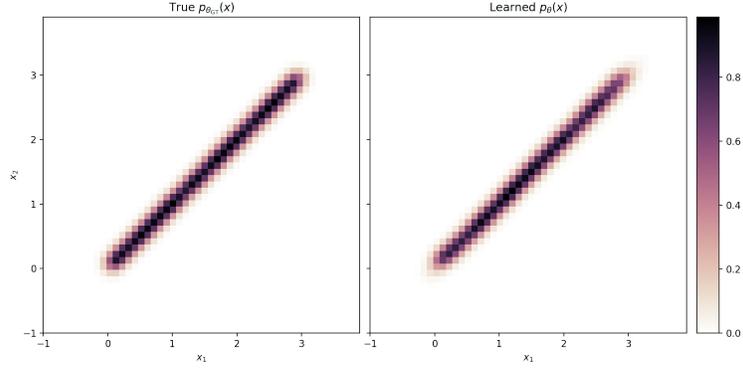


(d) Posteriors under true f_{θ}

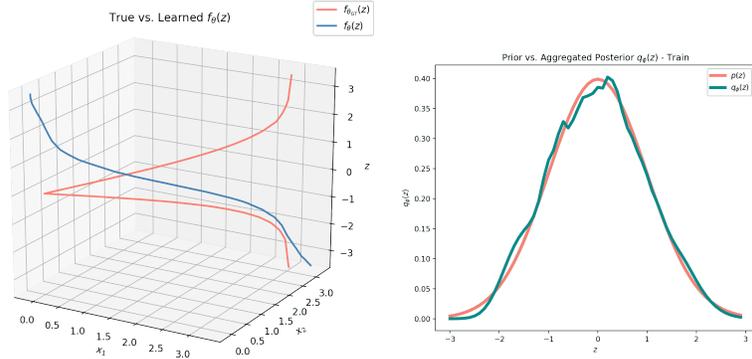


(e) Posteriors under learned f_{θ}

Figure 5: MFG-VAE trained on the Circle Example. In this toy data, condition (2) holds of Theorem 1 holds and condition (1) does not. To see this, notice that most examples of the posteriors are Gaussian-like, with the exception of the posteriors near $x = [1.0, 0.0]$, which are bimodal since in that neighborhood, x could have been generated using either $z > 3.0$ or using $z < -3.0$. Since only a few training points have a high posterior matching objective, a VAE is able to learn the data distribution well.

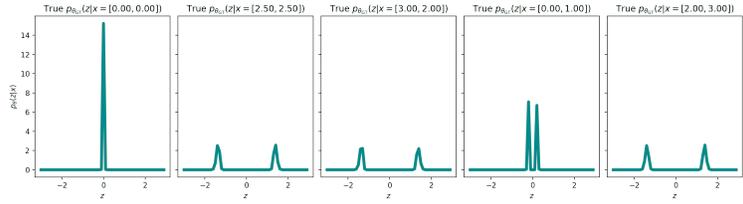


(a) True vs. learned $p_\theta(x)$

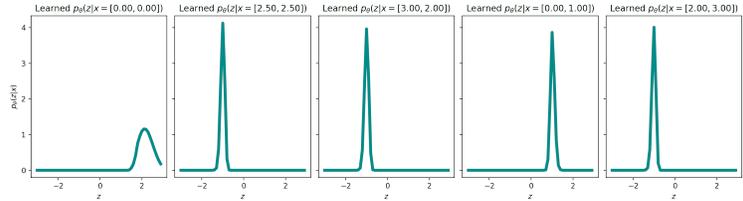


(b) True vs. learned $f_\theta(x)$

(c) Aggregated posterior vs. prior



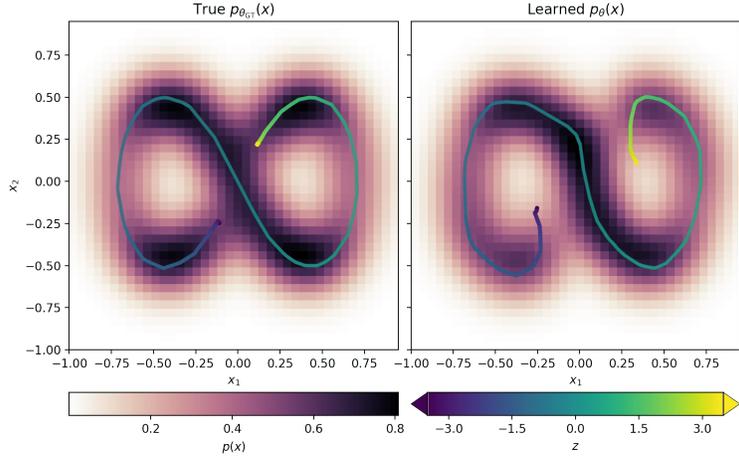
(d) Posteriors under true f_θ



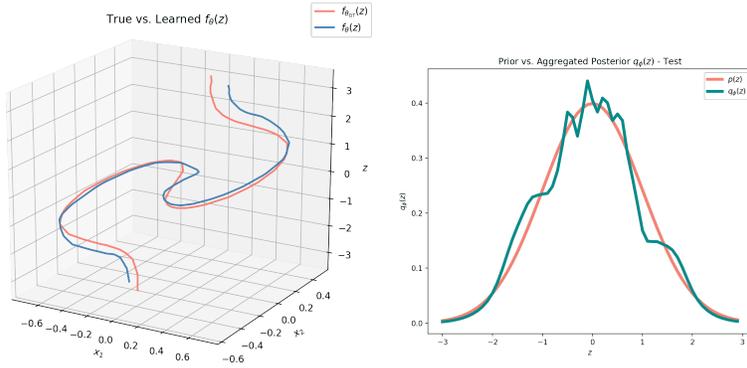
(e) Posteriors under learned f_θ

Figure 6: MFG-VAE trained on the Absolute-Value Example. In this toy data, condition (1) holds of Theorem 1 holds and condition (2) does not. To see this, notice that the function f_θ learned with a VAE is completely different than the ground-truth f_θ , and unlike the ground truth f_θ which has bimodal posteriors, the learned f_θ has unimodal posteriors (which are easier to approximate with a MFG). As such, a VAE is able to learn the data distribution well.

I Qualitative Demonstration of Unsupervised VAE Pathologies

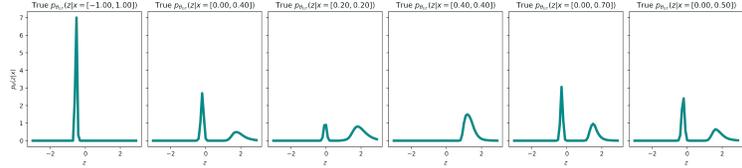


(a) True vs. learned $p_\theta(x)$, and learned vs. true $f_\theta(z)$, colored by the value of z .

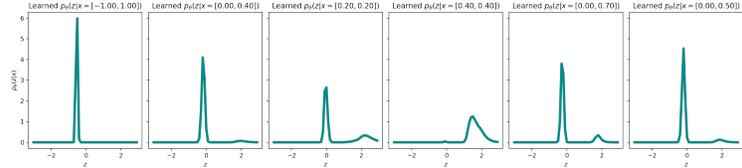


(b) True vs. learned $f_\theta(x)$

(c) Aggregated posterior vs. prior

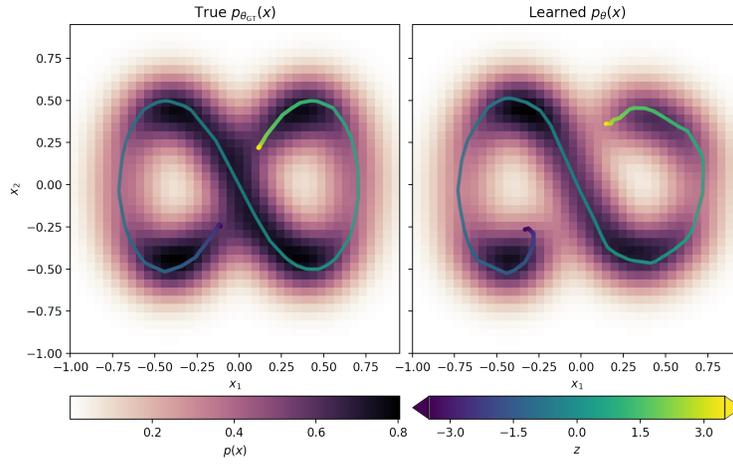


(d) Posters under true f_θ

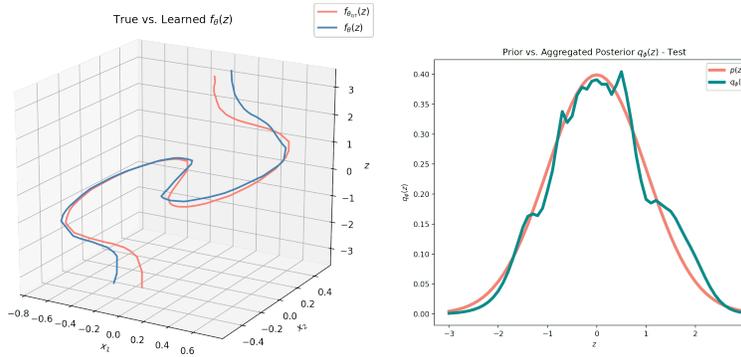


(e) Posters under learned f_θ

Figure 7: MFG-VAE trained on the Figure-8 Example. In this toy data, both conditions of Theorem 1 hold. The VAE learns a generative model with simpler posterior than that of the ground-truth, though it is unable to completely simplify the posterior as in the Absolute-Value Example. To learn a generative model with a simpler posterior, it curves the learned function f_θ at $z = -3.0$ and $z = 3.0$ away from the region where $z = 0$. This is because under the true generative model, the true posterior $p_\theta(z|x)$ in the neighborhood of $x \approx 0$ has modes around either $z = 0$ and $z = 3.0$, or around $z = 0$ and $z = -3.0$, leading to a high posterior matching objective.

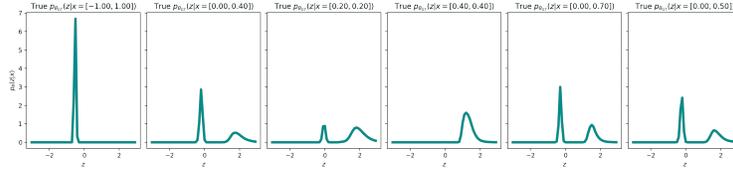


(a) True vs. learned $p_\theta(x)$, and learned vs. true $f_\theta(z)$, colored by the value of z .

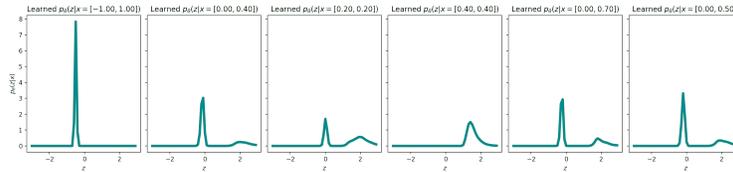


(b) True vs. learned $f_\theta(x)$

(c) Aggregated posterior vs. prior

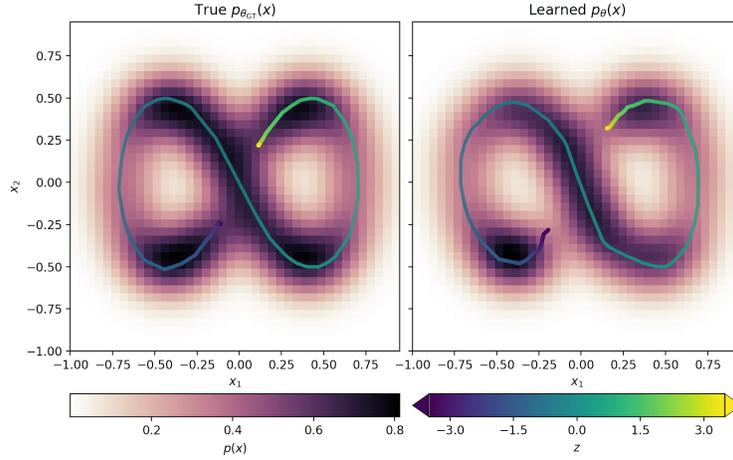


(d) Posters under true f_θ

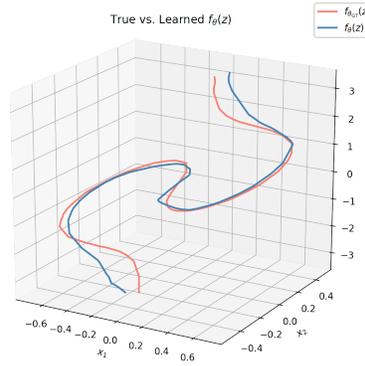


(e) Posters under learned f_θ

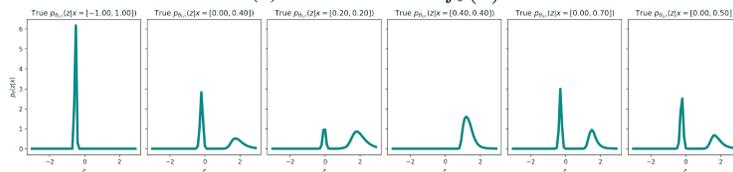
Figure 8: VAE with Lagging Inference Networks (LIN) trained on the Figure-8 Example. While LIN may help escape local optima, on this data, the training objective is still biased away from learning the true data distribution. As such, LIN fails in the same way a MFG-VAE does (see Figure 7).



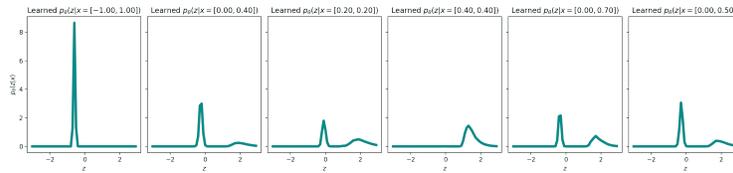
(a) True vs. learned $p_\theta(x)$, and learned vs. true $f_\theta(z)$, colored by the value of z .



(b) True vs. learned $f_\theta(x)$

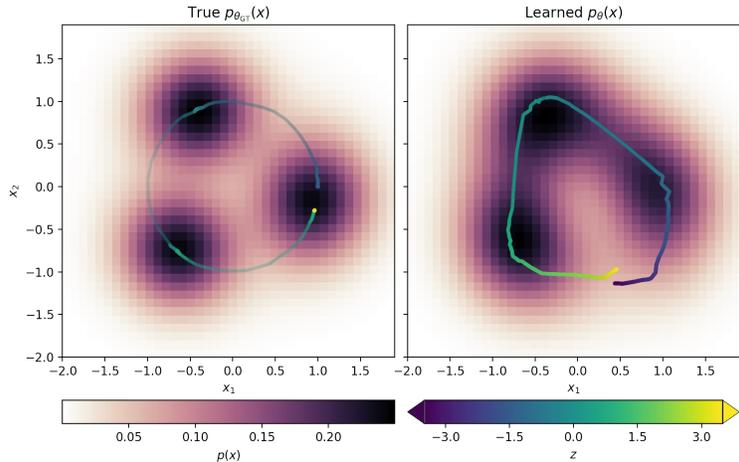


(c) Posteriors under true f_θ

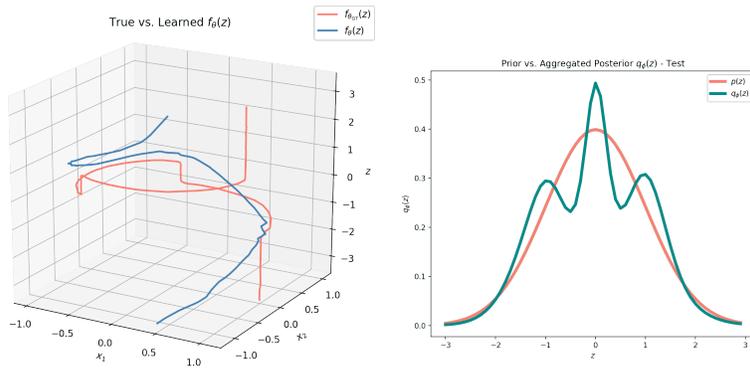


(d) Posteriors under learned f_θ

Figure 9: IWAE trained on the Figure-8 Example. In this toy data, both conditions of Theorem 1 hold. The IWAE learns a generative model with a slightly simpler posterior than that of the ground-truth. This is because even with the number of importance samples as large as $S = 20$, the variational family implied by the IWAE objective is not sufficiently expressive. The objective therefore prefers to learn a model with a lower data marginal likelihood. While increasing $S \rightarrow \infty$ will resolve this issue, it is not clear how large a S is necessary and whether the additional computational overhead is worth it.

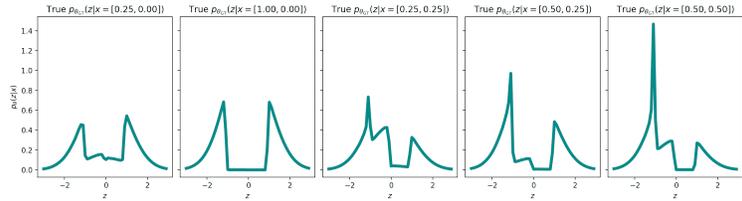


(a) True vs. learned $p_\theta(x)$, and learned vs. true $f_\theta(z)$, colored by the value of z .

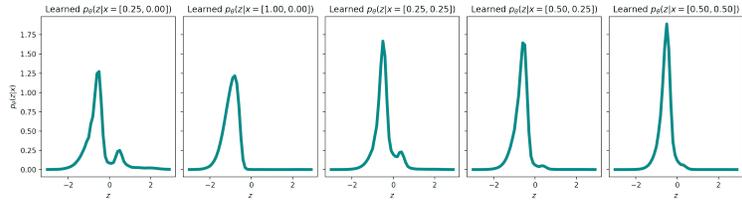


(b) True vs. learned $f_\theta(x)$

(c) Aggregated posterior vs. prior

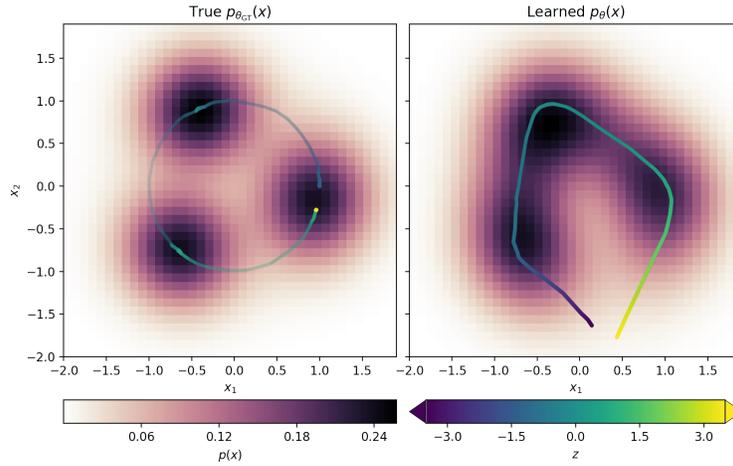


(d) Posters under true f_θ

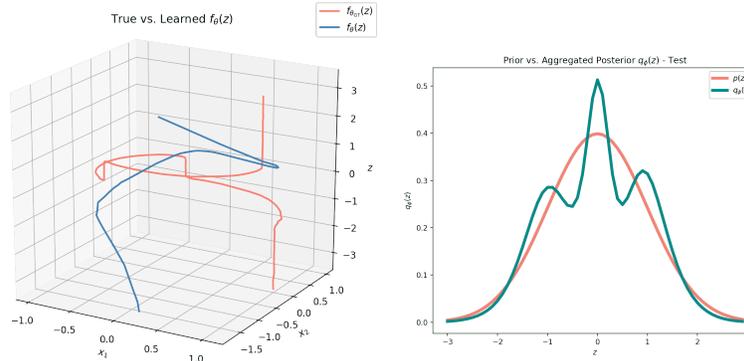


(e) Posters under learned f_θ

Figure 10: MFG-VAE trained on the Clusters Example. In this toy data, both conditions of Theorem 1 hold. The VAE learns a generative model with simpler posterior than that of the ground-truth, though it is unable to completely simplify the posterior as in the Absolute-Value Example. To learn a generative model with a simpler posterior, it learns a model with a function $f_\theta(z)$ that, unlike the ground truth function, does not have steep areas interleaved between flat areas. As such, the learned model is generally more flat, causing the learned density to be “smeared” between the modes.

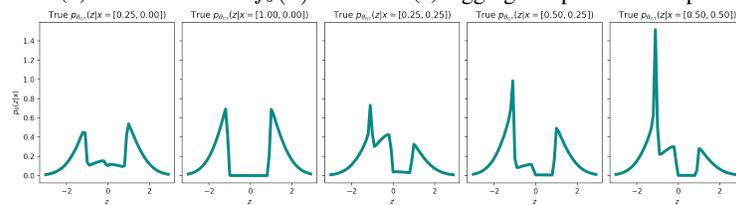


(a) True vs. learned $p_\theta(x)$, and learned vs. true $f_\theta(z)$, colored by the value of z .

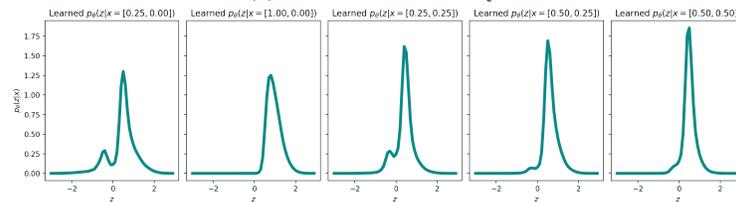


(b) True vs. learned $f_\theta(x)$

(c) Aggregated posterior vs. prior

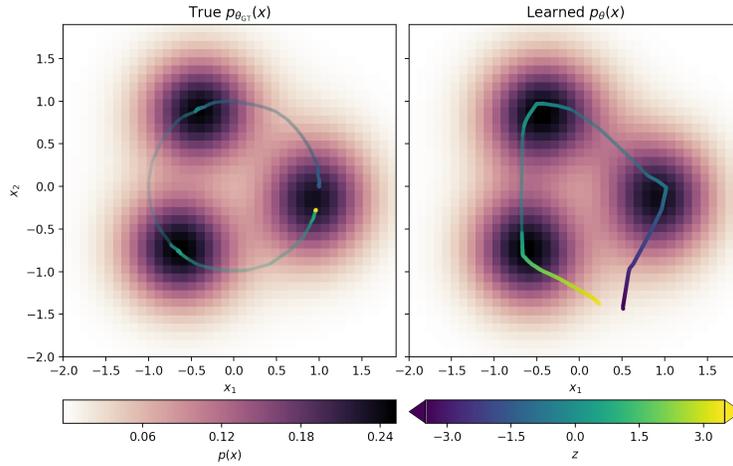


(d) Posteriors under true f_θ

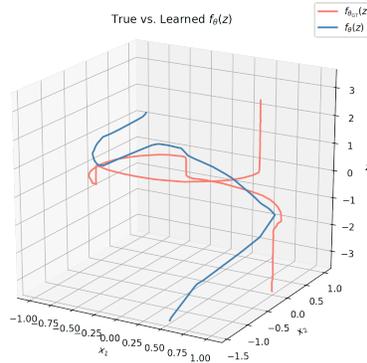


(e) Posteriors under learned f_θ

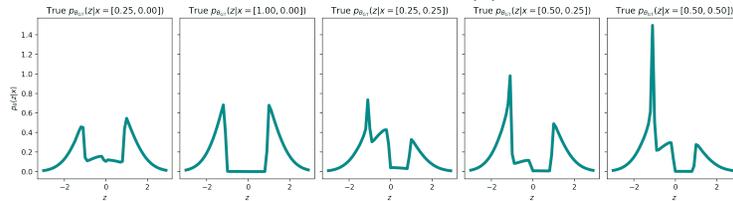
Figure 11: VAE with Lagging Inference Networks (LIN) trained on the Clusters Example. While LIN may help escape local optima, on this data, the training objective is still biased away from learning the true data distribution. As such, LIN fails in the same way a MFG-VAE does (see Figure 10).



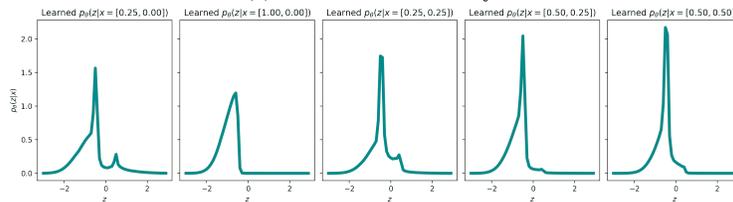
(a) True vs. learned $p_\theta(x)$, and learned vs. true $f_\theta(z)$, colored by the value of z .



(b) True vs. learned $f_\theta(x)$

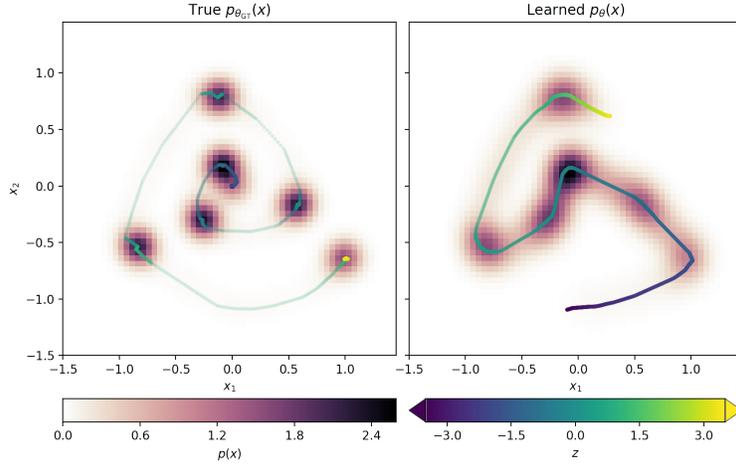


(c) Posteriors under true f_θ

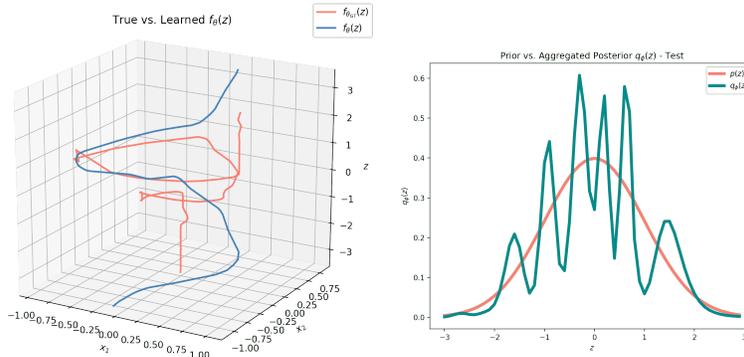


(d) Posteriors under learned f_θ

Figure 12: IWAE trained on the Clusters Example. In this toy data, both conditions of Theorem 1 hold. IWAE is able to learn the ground truth data distribution while finding a generative model with a simpler posterior than that of the ground-truth model.

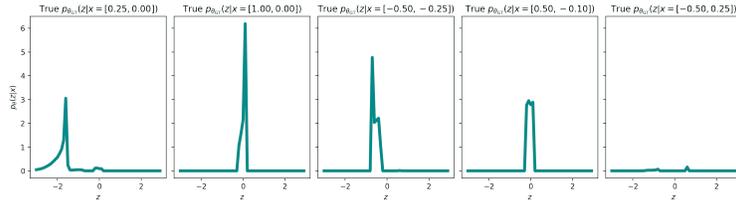


(a) True vs. learned $p_\theta(x)$, and learned vs. true $f_\theta(z)$, colored by the value of z .

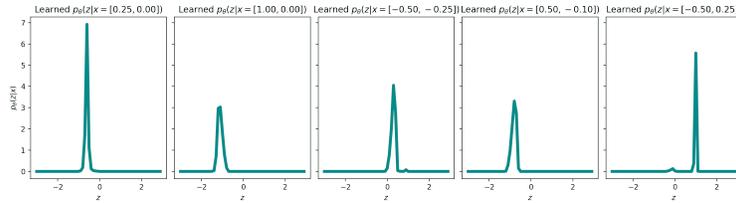


(b) True vs. learned $f_\theta(x)$

(c) Aggregated posterior vs. prior



(d) Posteriors under true f_θ



(e) Posteriors under learned f_θ

Figure 13: MFG-VAE trained on the Spiral-Dots Example jointly over $\theta, \phi, \epsilon_\epsilon^2$. In this toy data, as Theorem 2 predicts, the ELBO drastically misestimates the observation noise. The VAE learns a generative model with simpler posterior than that of the ground-truth, though it is unable to completely simplify the posterior as in the Absolute-Value Example. To learn a generative model with a simpler posterior, it learns a model with a function $f_\theta(z)$ that, unlike the ground truth function, does not have steep areas interleaved between flat areas. As such, the learned model is generally more flat, causing the learned density to be “smeared” between the modes. Moreover due to the error in approximating the true posterior with a MFG variational family, the ELBO misestimates σ_ϵ^2 .

J Qualitative Demonstration of Semi-Supervised VAE Pathologies

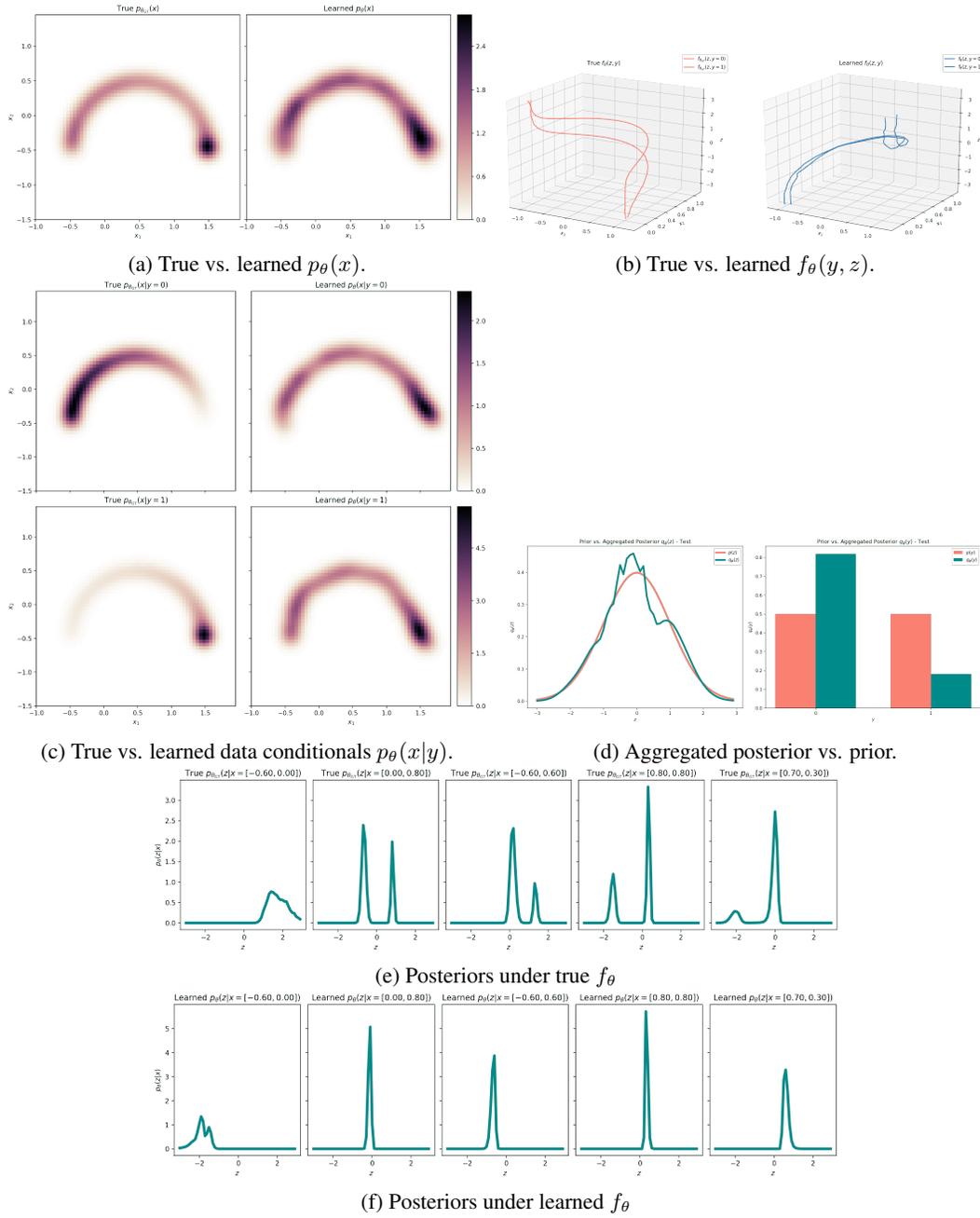
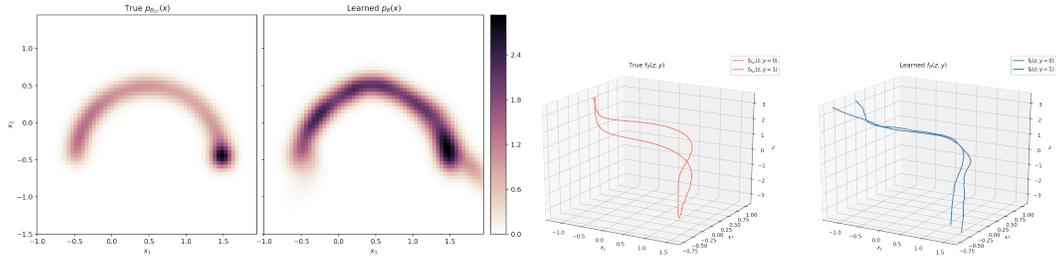
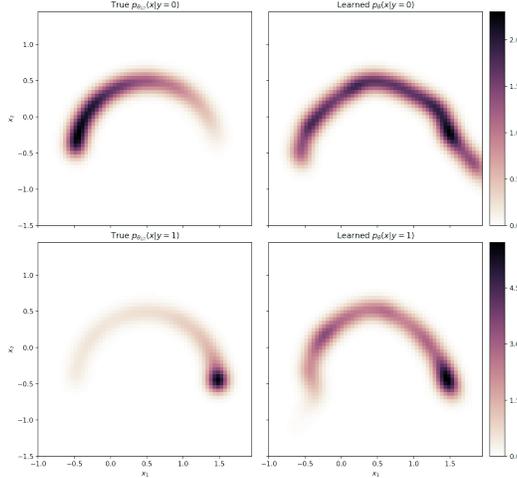


Figure 14: Semi-Supervised MFG-VAE trained on the Discrete Semi-Circle Example. While using semi-supervision, a VAE is still able to learn the $p(x)$ relatively well. However, in this example, given x there is uncertainty as to whether it was generated from $f_\theta(y = 0, z)$ or from $f_\theta(y = 1, z)$, the posterior $p_\theta(z|x)$ is bimodal and will cause a high posterior matching objective. Since semi-supervised VAE objective prefers models with simpler posteriors, the VAE learns a unimodal posterior by collapsing $f_\theta(y = 0, z) = f_\theta(y = 1, z)$, causing $p(x|y = 0) \approx p(x|y = 1) \approx p(x)$. The learned model will therefore generate poor sample quality counterfactuals.

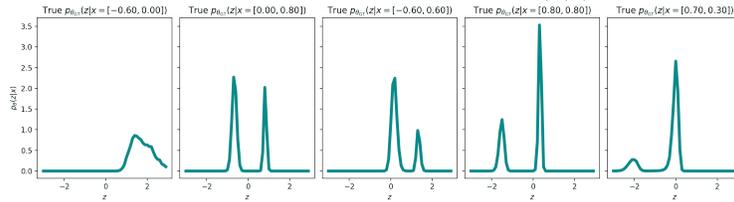


(a) True vs. learned $p_\theta(x)$.

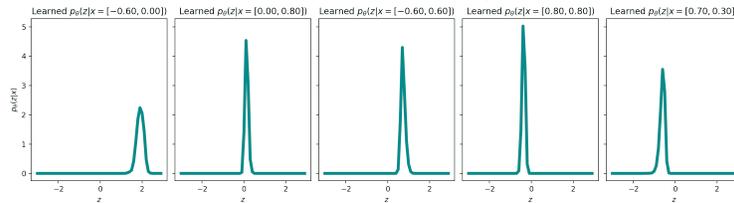
(b) True vs. learned $f_\theta(y, z)$.



(c) True vs. learned data conditionals $p_\theta(x|y)$.



(d) Posteriors under true f_θ



(e) Posteriors under learned f_θ

Figure 15: Semi-Supervised VAE trained with Lagging Inference Networks (LIN) trained on the Discrete Semi-Circle Example. While LIN may help escape local optima, on this data, the training objective is still biased away from learning the true data distribution. As such, LIN fails in the same way a MFG-VAE does (see Figure 14).

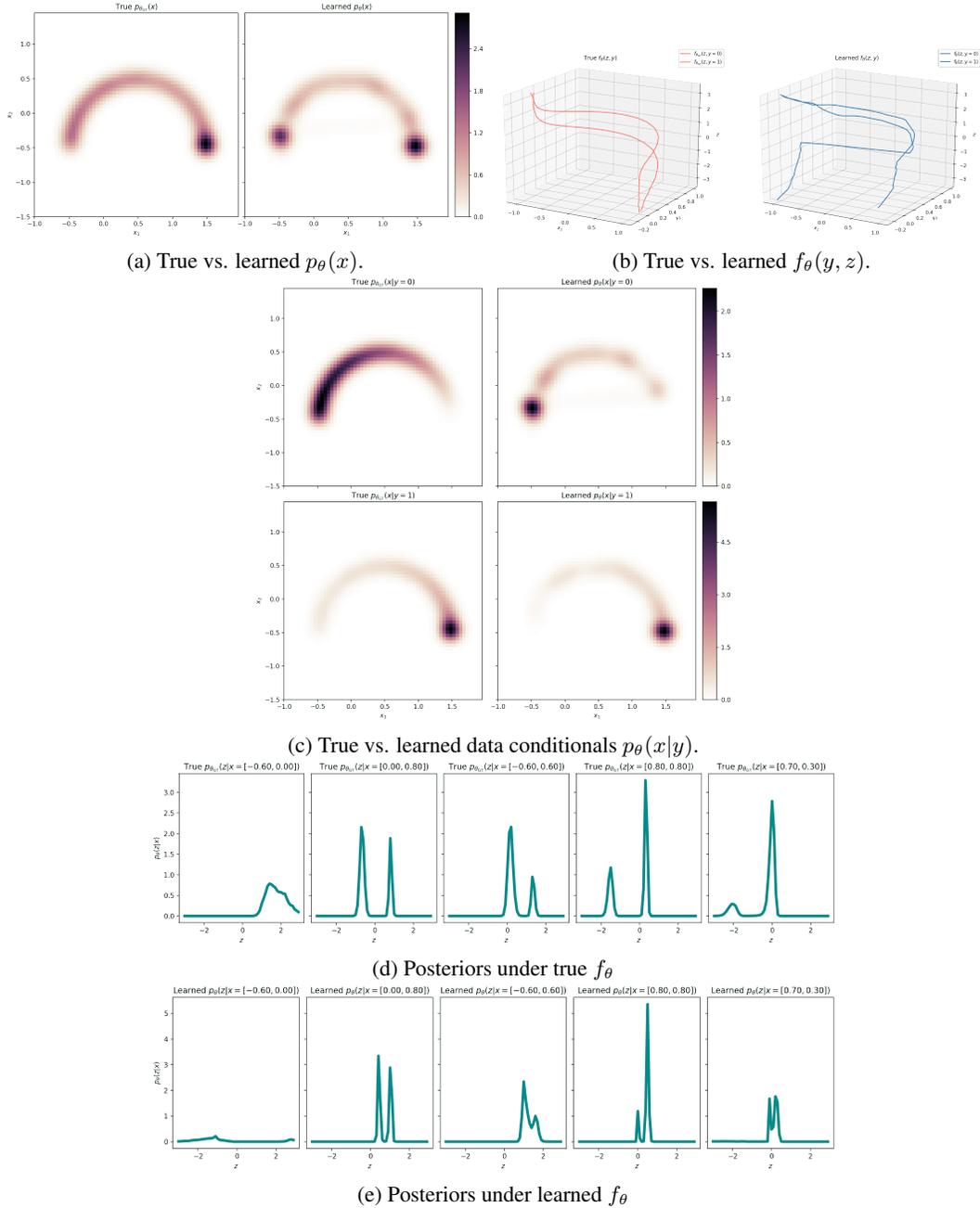


Figure 16: Semi-Supervised IWAE trained on the Discrete Semi-Circle Example. While using semi-supervision, a IWAE is still able to learn the $p(x)$ and $p(x|y)$ better than a VAE. This is because it allows for more complicated posteriors and therefore does not collapse $f_\theta(y = 0, z) = f_\theta(y = 1, z)$. However, since IWAE has a more complex variational family, the variational family no longer regularizes the function f_θ . As such, in order to put enough mass on the left-side of the semi-circle, f_θ jumps sharply from the right to the left, as opposed to preferring a simpler function such as the ground truth function.

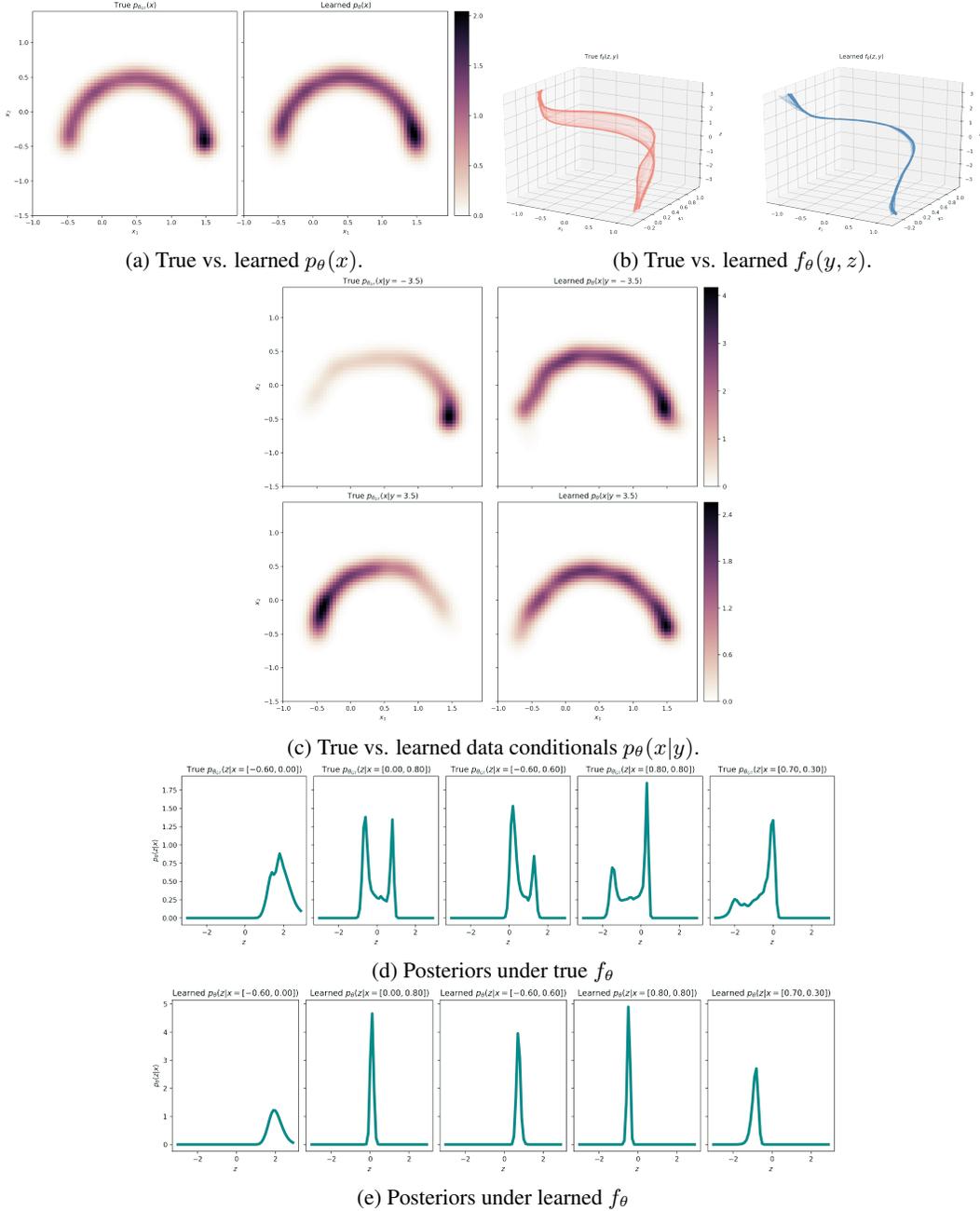
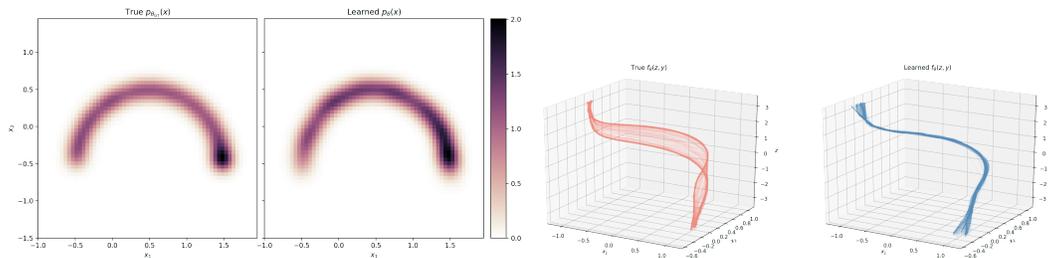
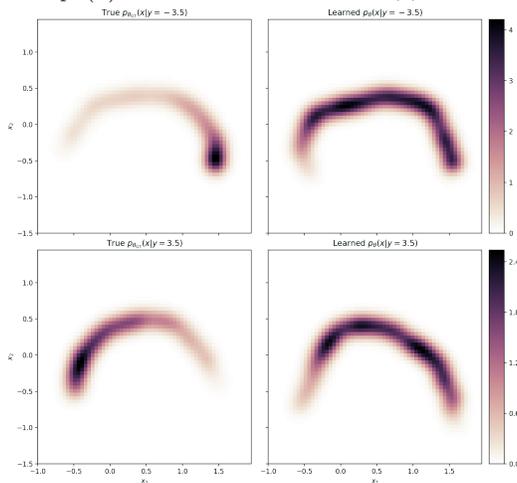


Figure 17: Semi-Supervised MFG-VAE trained on the Continuous Semi-Circle Example. In this example, the VAE exhibits the same problems as in the Discrete Semi-Circle Example (Figure 17). However, since y is continuous, this poses an additional issue. Since $q_\phi(y|x)$ (the discriminator) in the objective is a Gaussian, and the ground truth $p_\theta(y|x)$ is multi-modal, the objective will select a function f_θ under which $p_\theta(y|x)$ is a MFG. This, again, leads to learning a model in which $f_\theta(y = \cdot, z)$ are the same for all values of y , causing $p(x|y = 0) \approx p(x|y = 1) \approx p(x)$. The learned model will therefore generate poor sample quality counterfactuals.

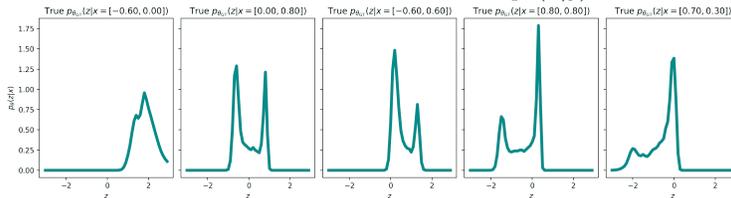


(a) True vs. learned $p_\theta(x)$.

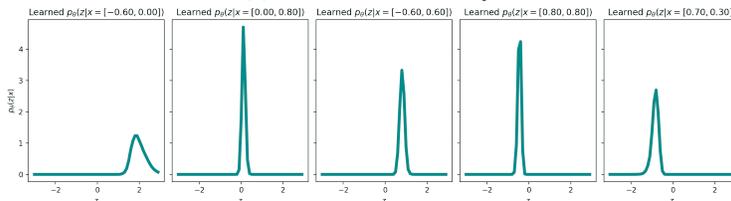
(b) True vs. learned $f_\theta(y, z)$.



(c) True vs. learned data conditionals $p_\theta(x|y)$.

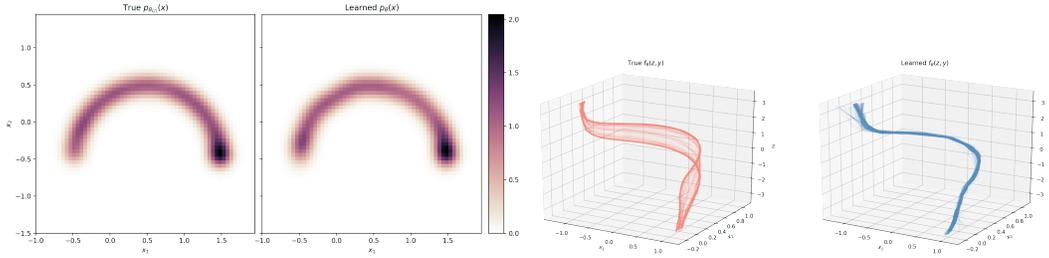


(d) Posteriors under true f_θ



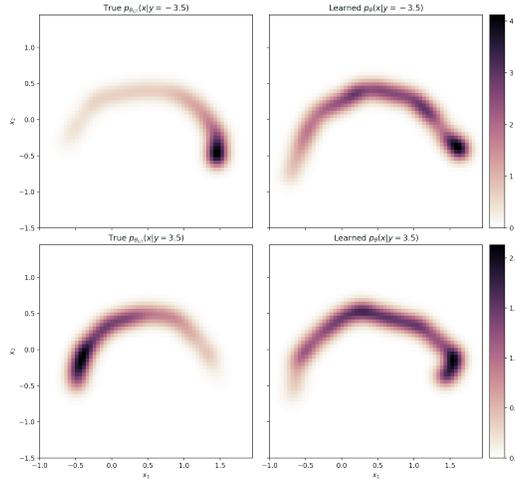
(e) Posteriors under learned f_θ

Figure 18: Semi-Supervised VAE trained with Lagging Inference Networks (LIN) trained on the Continuous Semi-Circle Example. While LIN may help escape local optima, on this data, the training objective is still biased away from learning the true data distribution. As such, LIN fails in the same way a MFG-VAE does (see Figure 17).

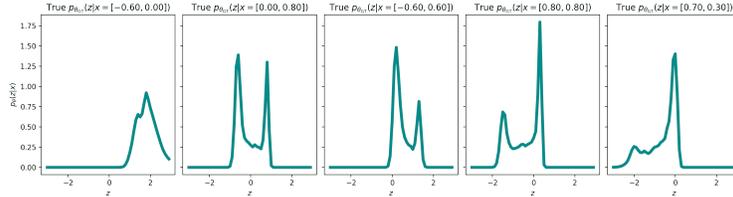


(a) True vs. learned $p_{\theta}(x)$.

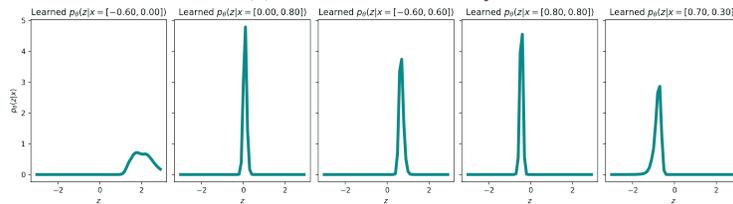
(b) True vs. learned $f_{\theta}(y, z)$.



(c) True vs. learned data conditionals $p_{\theta}(x|y)$.



(d) Posteriors under true f_{θ}



(e) Posteriors under learned f_{θ}

Figure 19: Semi-Supervised IWAE trained on the Continuous Semi-Circle Example. While using semi-supervision, a IWAE is still able to learn the $p(x)$ and $p(x|y)$ better than a VAE. However, since $q_{\phi}(y|x)$ (the discriminator) in the objective is a Gaussian, and the ground truth $p_{\theta}(y|x)$ is multi-modal, the objective will select a function f_{θ} under which $p_{\theta}(y|x)$ is a MFG. This, again, leads to learning a model in which $f_{\theta}(y = \cdot, z)$ are the same for all values of y , causing $p(x|y = 0) \approx p(x|y = 1) \approx p(x)$. The learned model will therefore generate poor sample quality counterfactuals.

K Posterior Is Simpler Under Model Mismatch

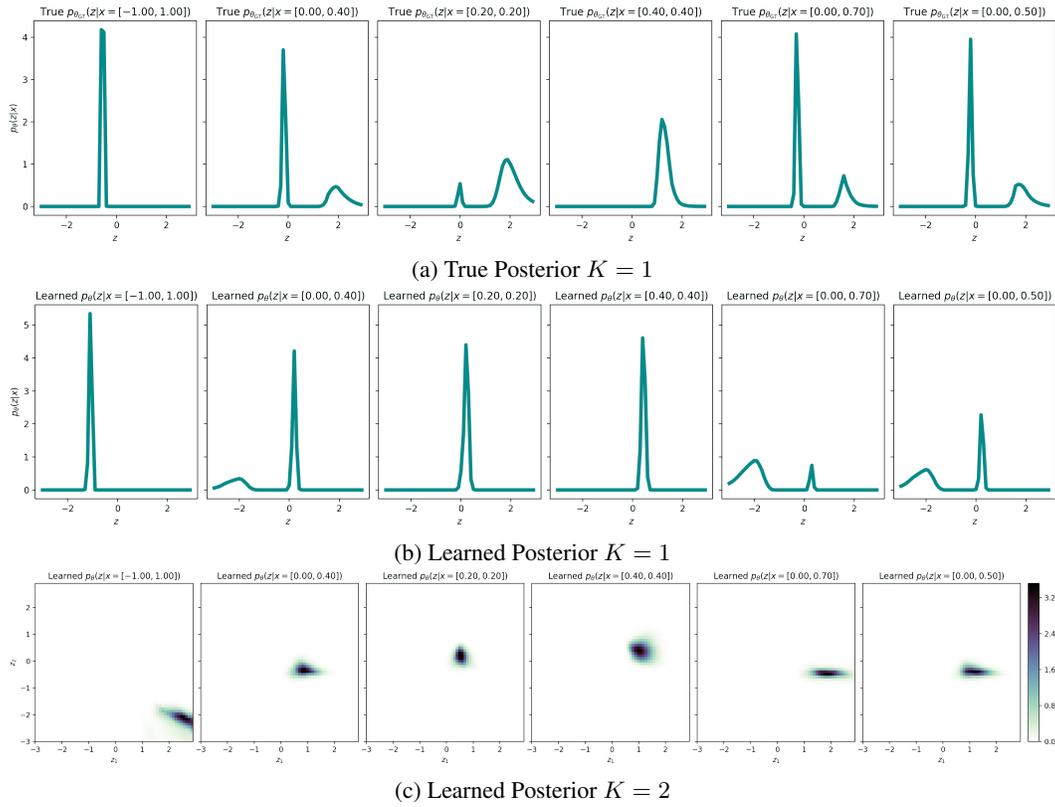


Figure 20: MFG-VAEs learn simpler posteriors as latent dimensionality K increases and as the observation noise σ_ϵ^2 decreases on “Figure-8 Example” (projected into 5D space). The figure shows that on 5 selected points, the true posterior for the learned model becomes simpler as K increases.